Bends, Jogs, And Wiggles for Railroad Tracks and Vehicle Guide Ways

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ABSTRACT

In traditional design for railroad track other than "special work" the geometrical path of a track is composed as a sequence of straight lines, circular arcs, and spirals. The present paper defines some new geometrical shapes that can be incorporated into alignments for railroad tracks and other vehicle guide ways. Where the new shapes are appropriate, alignments that incorporate them will give dynamic performance superior to corresponding alignments whose only curved elements are spirals and circular arcs.

The simplest of the new shapes are referred to as Bends, Jogs, and Wiggles. While these shapes have been a part of every-day language for a long time, they do not appear to have been previously defined for or used in geometrical design of tracks and guide ways.

The new shapes are defined within the framework of a recently developed method for design of improved railroad track spirals. This paper reviews that method, notes situations in which the new shapes can provide improved geometry, presents mathematical formulae by which the shapes can be defined, and shows by examples how the shapes can be applied in some typical railroad track situations. It is shown that the new shapes can be defined in terms of Gegenbauer polynomials and that the known properties of those polynomials contribute both to understanding and application of the shapes.

Among ways that the new shapes can be used to improve railroad track there are three that are particularly encouraged. First, the Jog shape can be used to define turnout and crossover geometry that is dynamically better than the geometry in current use. Second, when existing curves are being re-aligned and their spirals are found to be too distorted for immediate restoration of ideal geometry, the spirals can be modified by admixture of the new shapes and tamping with limited track throws can then achieve smoothed alignments whose dynamic characteristics are relatively optimal. Third, when existing curves are being upgraded for higher speeds and transition lengths need to be increased without relocation of the curves, transitions based on combinations of spirals and Bends will be dynamically better than corresponding compound transitions based on separate arcs and spirals.

The new shapes are applicable not only to railroad tracks but also to other vehicle guide ways including, for example, maglev guide ways (magways), roller coaster tracks, and bobsled runs.

OUTLINE

1. Introduction

Recent publications (references 1, 2, & 3) have presented and demonstrated an improved way of thinking about and calculating the geometry of transition spirals for railroad curves. Those publications applied the improved method to the most common situation where a spiral makes a simple connection between two adjacent segments of track with differing constant values of curvature.

The present paper explores the application of the improved method to more complex transition shapes referred to as Bends, Jogs, and Wiggles.

Bends will be of value for track layout at locations where the track needs to change direction by a “small” amount. The background is as follows. When an attempt is made to use the normal spiral – curve – spiral sequence to accomplish a “small” turn, the spirals themselves accomplish the turn so that the sequence changes to spiral – spiral. If the spirals employed are traditional linear spirals, then the two spiral sequence has worse than usual dynamic characteristics where the two spirals meet. That problem can be ameliorated by use of improved spirals. However, it will be more logical to employ a shape designed for this situation. The Bend is designed to be such a shape. It is defined using a conceptual framework first developed for improving the design of spirals and will give good dynamic performance in “small turn” situations.

Jogs are defined below for situations where the track needs to curve quickly in one direction and then quickly in the other direction and where the curvature of the track needs to keep changing throughout both curves. Jogs can be used for the design of turnouts and crossovers. A Jog can also be used in continuous track where a clearance obstruction on one side is followed fairly closely by an obstruction on the other side.

A **Wiggle** bends to the right, then to the left, and then to the right, or vice-versa. A Wiggle can be used where an obstacle on one side requires a local deviation by what would otherwise be a straight path. Shapes like Wiggles but with more than three bends can also be defined.

This paper gives formulae for Bends, Jogs, and Wiggles and illustrates track shapes which can be obtained from them. The author expects to be able in a future paper coauthored with others to present results of simulations like those of reference 3 that will show how predicted vehicle dynamic responses to Bends, Jogs, and shapes defined by Gegenbauer polynomial series compare with predicted vehicle responses to corresponding traditional geometries.

The new shapes are applicable to any guide ways on which vehicle speeds are high enough relative to guide way curvatures so that centripetal accelerations are important. Other examples of guide ways in that category are guide ways for maglev vehicles, roller coasters tracks, and bob-sled runs.

2. Review of Improved Spiral Design Method

The present paper is based on an improved method for designing track geometry shapes in which the curvature varies with distance. This Section gives an overview of that method as applied to the design of the spiral, which is the simplest such shape.

The main novelty of the improved method is that design is begun not by consideration of competing shapes but rather by considering competing forms for the roll of the track as a function of distance. The rationale for the improved design method is the proposition that the primary job of a shape element in which curvature changes with distance is to cause a vehicle that traverses it to have its roll angle change from one steady value to another with the least fluctuation of lateral force applied to the rails and with the least fluctuation of lateral and roll accelerations applied to vehicles. This premise focuses attention on the vehicle's rotation about its roll axis as it traverses the spiral and on the character of the roll and linear accelerations to which the vehicle is subjected in that process. For reasons explained in references 1, 2, 3, 4, and 5 it is normally advantageous to have the longitudinal axis for roll of the track raised above the plane of the track to the height of a typical vehicle center of gravity or higher.

Within the improved method, after a roll motion has been chosen, there is a need to be able to compute the shape that corresponds to that roll motion. The computation begins with the generally accepted premise that the curvature of the path of the roll axis should be such that at design speed the centripetal acceleration due to the speed and curvature at any given point balances the component of gravity due to the roll (i.e., bank or superelevation angle) of the track at that point. Looking at the components of centripetal acceleration and gravity in the rolled plane of the track, this premise is expressed by the differential equation

$$\frac{d}{ds} b\_axis(s) = \left(\frac{g}{v^2}\right) \tan\left(r\_angle(s)\right) \tag{1}$$

In equation (1) *b_axis(s)* denotes the bearing angle of the path of the roll axis. Its derivative with respect to distance along the path of the roll axis is by definition the curvature of that path. g is the acceleration of gravity. v is the vehicle speed for which the gravitational and centripetal acceleration components are to be in balance, and *r_angle(s)* is the roll angle of the track as a function of distance along the path of the roll axis.

Once a roll motion is selected so that track roll is specified as a function of distance, integrating equation (1) once yields the bearing angle of the roll axis as a function of distance, and integrating the sine and cosine of the roll axis path bearing angle with respect to distance yields Cartesian coordinates of points on the path of the roll axis as functions of distance along it. With the roll angle of the track and the path of the roll axis both known as functions of distance along the path of the roll axis, the path of the track itself can be inferred by simple trigonometry as illustrated in Figure 1.

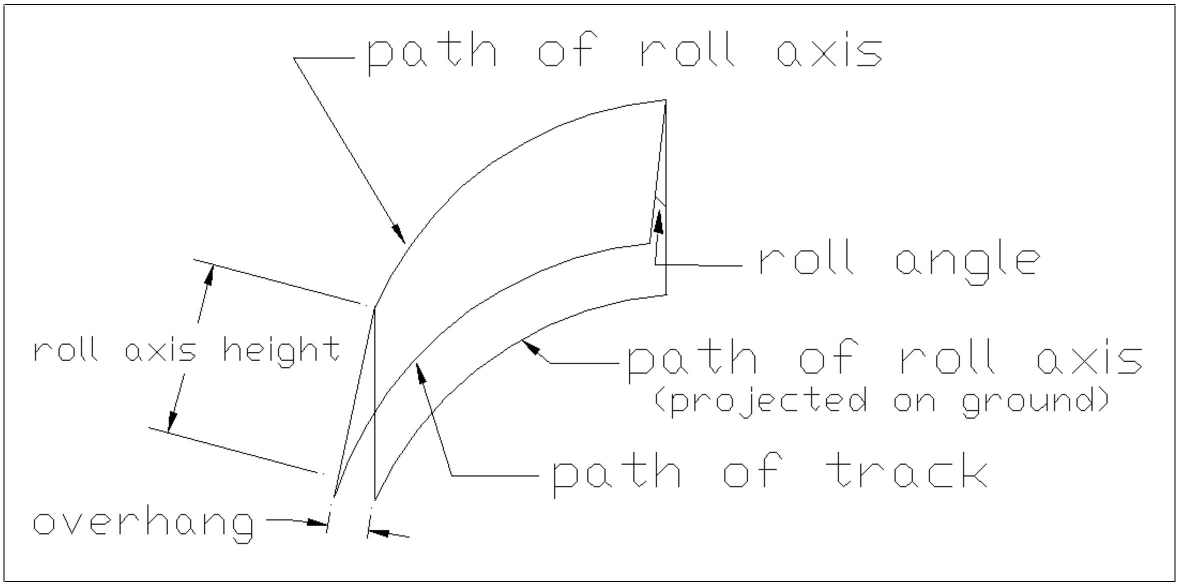

FIGURE 1. Illustration of elevation of roll axis height above plane of track.

A track shape obtained as just outlined based on a roll motion selected as an initial guess will be unlikely to connect properly to the adjacent track segments that the shape is intended to connect. The method becomes practical when the roll motion is defined by formulae that have adjustable parameters and when there is a computational procedure by which the parameter values can be adjusted so that the resulting shape does connect properly with the segments of track that are adjacent to it. References 2 and 3 explain the computational procedure for obtaining improved spirals and give examples of plausible roll motions and of the spiral shapes that they generate. The computational procedure is also demonstrated in Section 10 below.

The following example of a roll motion for a spiral and of the spiral shape that the roll motion generates is taken from reference 3. The shapes of the roll functions are illustrated in Figure 2.

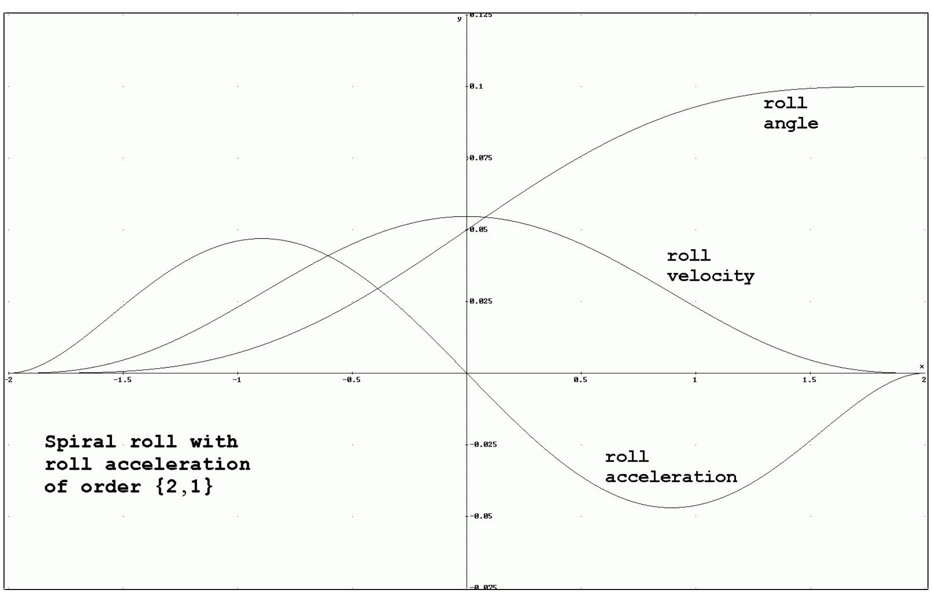

FIGURE 2 Order {2,1} roll function; the roll acceleration has a simple zero at the mid point and a 2nd order zero at each end.

The roll angle and each of its derivatives is given by a single function that applies throughout the length of the spiral. The roll function shown is referred as order {2,1}. The {2,1} designation is used to indicate that the roll acceleration has a 2nd order zero at each end and a 1st order zero at the midpoint. Denote distance along the path of the roll axis of the spiral by s, let the spiral extend from a distance $s = -a$ to a distance $s = a$, and let roll_change denote the change in the track roll angle over the length of the spiral. Then the formula for the roll acceleration (meaning the 2nd derivative of track roll angle in radians with respect to distance along the path of the roll axis) is

$$\frac{d^2}{ds^2} r\_angle(s) = \frac{-105 \cdot roll\_change \cdot (a+s)^2 \cdot s \cdot (a-s)^2}{16\, a^7} \tag{2}$$

Figure 3 illustrates the spiral that is obtained from the above roll function for connecting tangent track to a curve with curvature of 1.0 deg per 100 ft chord and that is "offset" from the tangent by 1.8179 ft. The track roll axis is at a height 2.44 m (8 ft) above the (unrolled) plane of the track. In this figure the improved spiral obtained from the above roll function is compared with a traditional linear spiral connecting the same tangent and curve. It may be seen that the improved spiral is a little over twice as long as the traditional spiral and that it is smoother in character than the linear spiral in the vicinity of the end points of the latter. (The dynamic disturbance caused by the linear spiral would be reduced if the length of the linear spiral were increased, but such an increase would cause an increase in the offset between the spiral and the curve and would require relocation of the curve.)

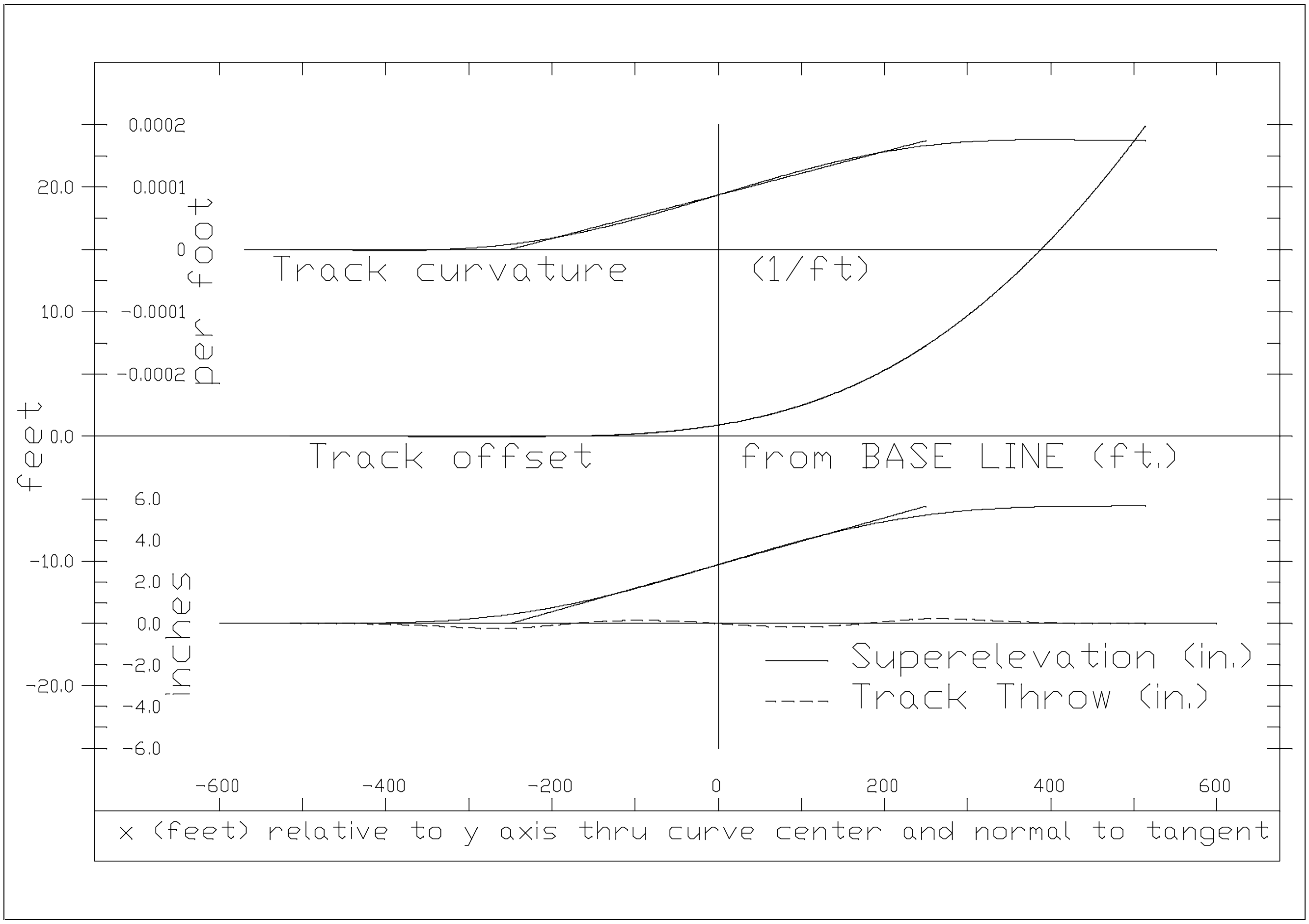

FIGURE 3 Plots of curvature, alignment, and superelevation for an order {2,1} spiral with 2.44 m (8 ft) roll axis height and for a corresponding traditional linear spiral. Both spirals connect tangent track to a 1.0 deg curve elevated for balance at 90 mph. (The two alignments shown in the central part of the plot are too close together to be distinguished at this scale. The displacement from one to the other is indicated by the track throw.) The lengths of the traditional and improved spirals illustrated are 152.4 m (500 ft) and 313.6 m (1028.8 ft) respectively.

This Section concludes with two observations about the behavior of the track curvature.

The first observation pertains to the behavior of the track curvature at the ends of a track shape. As explained in reference 2, rail vehicle motion simulations have shown that a discontinuity in the first derivative of track curvature can excite episodes of hunting on the part of some trucks. We would therefore like to see how to insure that the first derivative of track curvature will be continuous (which is to say, zero) at each end of a track shape.

In light of Figure 1, at corresponding points on the track and on the path of the roll axis, the compass bearing of the track is related to the compass bearing of the path of the roll axis by the formula

$$b\_track(s) = b\_axis(s) - \arctan\left(h \cdot \frac{d}{ds} \sin\bigl(r\_angle(s)\bigr) \right) \tag{3}$$

where h represents the roll axis height. The formula for the curvature of the track shape as a function of distance along the path of the roll axis can be obtained by differentiating equation (3) with respect to distance along the track. Denoting distance along the track by z and for the moment abbreviating *r_angle(s)* as *r(s)*, that curvature is

$$\frac{d}{dz} b\_track(s) = \frac{d}{ds} b\_track(s) \cdot \frac{ds}{dz} = \frac{ds}{dz} \cdot \left\{ \frac{d}{ds} b\_axis(s) - \frac{h \cdot \frac{d^2}{ds^2} \sin(r(s))}{1 + \left[h \cdot \frac{d}{ds} \sin(r(s)) \right]^2} \right\} \tag{4}$$

Looking further at Figure 1 one can obtain the relation

$$\frac{dz}{ds} = \sqrt{\left(1 + h \cdot \sin(r(s)) \cdot \frac{d}{ds} b\_axis(s)\right)^2 + \left(h \cdot \cos(r(s)) \cdot \frac{d}{ds} r(s)\right)^2} \tag{5}$$

Our purpose here is to establish a condition under which the first derivative of the track curvature will be zero at the ends of the shape. It is apparent that $\frac{ds}{dz} = \left(\frac{dz}{ds}\right)^{-1}$ will always be close to unity so that we can ignore that factor and look just at how to insure that $\frac{d^2}{ds^2} b\_track(s)$, the 2nd derivative of the track bearing with respect to distance along the path of the roll axis, will be zero at the ends of a shape. Taking advantage of equation (1) we can write

$$\frac{d}{ds} b\_track(s) = \left(\frac{g}{v^2}\right) \tan(r(s)) - \frac{h \cdot \frac{d^2}{ds^2} \sin(r(s))}{1 + \left[h \cdot \frac{d}{ds} \sin(r(s)) \right]^2} \tag{6}$$

Differentiating once more we obtain

$$\frac{d^2}{ds^2} b\_track(s) = \left(\frac{g}{v^2}\right) \cdot \frac{\frac{d}{ds} r(s)}{\cos^2(r(s))} + \frac{\left[h \cdot \frac{d^2}{ds^2} \sin(r(s)) \right]^2}{\left[1 + \left[h \cdot \frac{d}{ds} \sin(r(s)) \right]^2 \right]^2} - \frac{h \cdot \frac{d^3}{ds^3} \sin(r(s))}{\left[1 + \left[h \cdot \frac{d}{ds} \sin(r(s)) \right]^2 \right]} \tag{7}$$

Coupling equation (7) with the simple formula

$$\frac{d^3}{ds^3} \sin(r(s)) = -\sin(r(s)) \cdot \frac{d}{ds} r(s) \cdot \frac{d^2}{ds^2} r(s) + \cos(r(s)) \cdot \frac{d^3}{ds^3} r(s) \tag{8}$$

two things can be observed. First, if the roll axis is not raised above the plane of the track so that $h = 0$, then the first derivative of the track curvature with respect to distance will be zero at the ends of a shape if the roll velocity, $\frac{d}{ds} r(s)$, is zero there. Second if the roll axis is raised above the plane of the track so that $h > 0$, then to insure that the first derivative of the track curvature with respect to distance will be zero at the ends of a shape we will need to restrict considerations to roll functions for which the angular velocity, angular acceleration, and angular jerk, namely $\frac{d}{ds}r(s)$, $\frac{d^2}{ds^2}r(s)$, and $\frac{d^3}{ds^3}r(s)$, are all zero there.

In the roll acceleration of equation (2) the 2nd order zero at each end causes the angular jerk to be zero at each end, and this feature makes that roll acceleration suitable for use with the roll axis raised above the plane of the track. In Section 7 below we will look later at a situation in which it does not appear practical to raise the roll axis above the plane of the track. In that situation we will look at a roll acceleration function that has only a 1st order zero at each end.

The second observation regarding the behavoir of the track curvature is for some transition shapes there are regions in which it will be greater than the curvature of the path of the roll axis. When that is the case it may appear that the balance between centripetal and gravitational force components underlying equation (1) is not being realized. We therefore note that the call in equation (1) for balance based on the curvature of the path of the roll axis rather than on the curvature of the track is deliberate in relation to what is located at the height to which the roll axis is raised when that is a vehicle center of gravity or the shoulder of an typical seated passenger.

3. Constructing the roll acceleration for a Bend by overlapping two spirals.

As noted above, if between two segments of tangent track there is need for a "small" turn, then in traditional practice the turn is accomplished by placing two spirals back to back. A turn formed in this way has suboptimal dynamic characteristics, especially if the spirals in question are traditional linear spirals. If attention is focused on the roll motion of a vehicle through the turn, then it makes sense to look for a single roll acceleration function that covers the whole turn. An alignment that provides a transition between two non parallel tangents and that is obtained from a continuously varying roll acceleration function that is symmetric about its mid point is referred to herein as a Bend. We will look at two different ways of forming roll acceleration functions for Bends.

The first way takes the roll acceleration functions of two spirals like those just illustrated with one raising the curvature and the other lowering it, and positions them so that they partially overlap. As the constituent roll accelerations are type {2,1} we will denote the combination as 2-{2,1}.

The two roll acceleration functions being combined will have opposite signs and will apply in different ranges of distance along the track. We combine them with the help of the auxiliary function BOX(a,s,b) defined as 1 if a <= s <= b and 0 otherwise. A single roll acceleration function multiplied by BOX(a,s,b) will contribute only in the range of s values in which it applies. We move one roll acceleration function backward a distance q from the center of the bend and move the other forward from the center by the same amount. This will give a Bend with a length of $2(q + a)$. We write the formula for the sum of the roll accelerations as

$$\begin{aligned} d^2r/ds^2 = \; & - \mathrm{BOX}(-a - q, s, a - q)\cdot j\cdot(s + q)\,(s + q - a)^2\,(s + q + a)^2 \\ & + \mathrm{BOX}(-a + q, s, a + q)\cdot j\cdot(s - q)\,(s - q - a)^2\,(s - q + a)^2 \end{aligned} \tag{9}$$

The overall factor of j can be replaced by an expression proportional to the maximum magnitude of the roll angle, which occurs in the middle of the Bend.

Figures 4, 5, and 6 illustrate the shapes of the roll motions defined by the forgoing equation when a is set to 1.0, the maximum roll is set to 0.1 radians, and q is set successively to $0.1*a$, $0.7*a$, and $0.9*a$

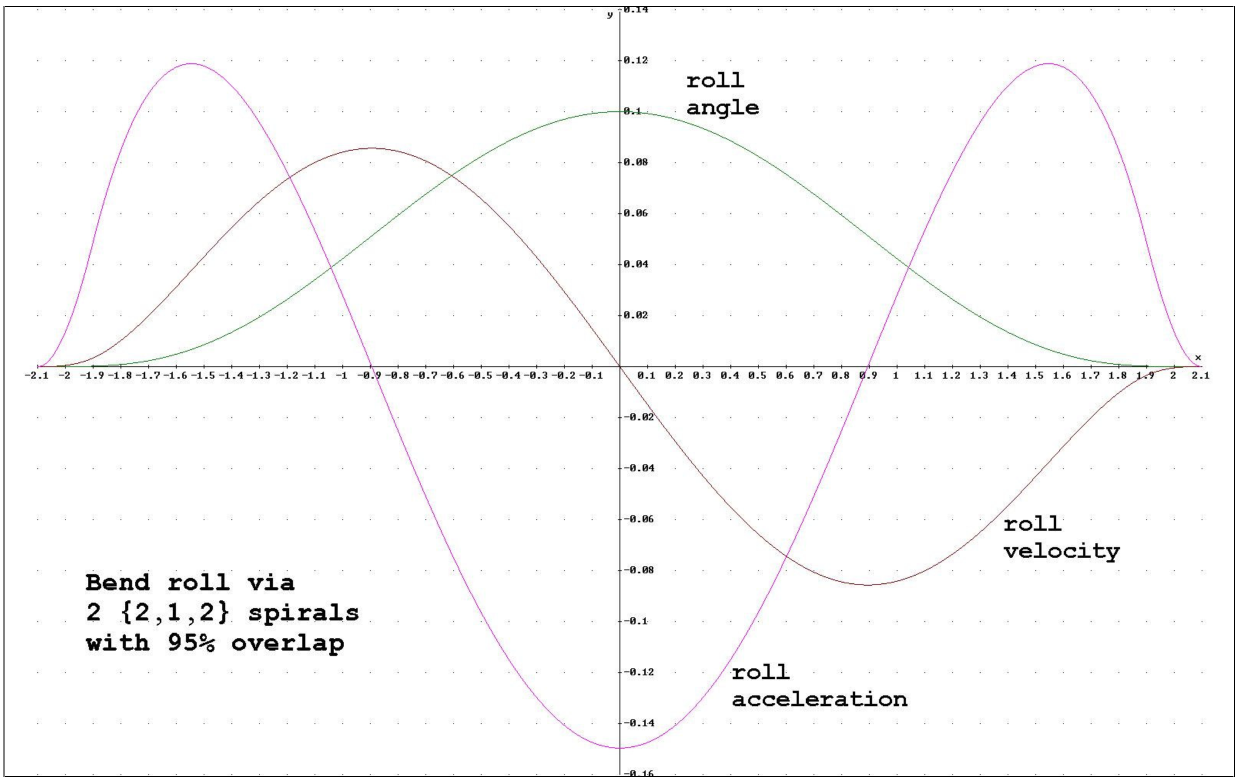

Figure 4. Roll functions for 2-{2,1} Bend with q = 0.1*a. (In this and following figures that show roll acceleration, velocity, and angle together the length of the shape is made artificially small so that the three curves have comparable heights.)

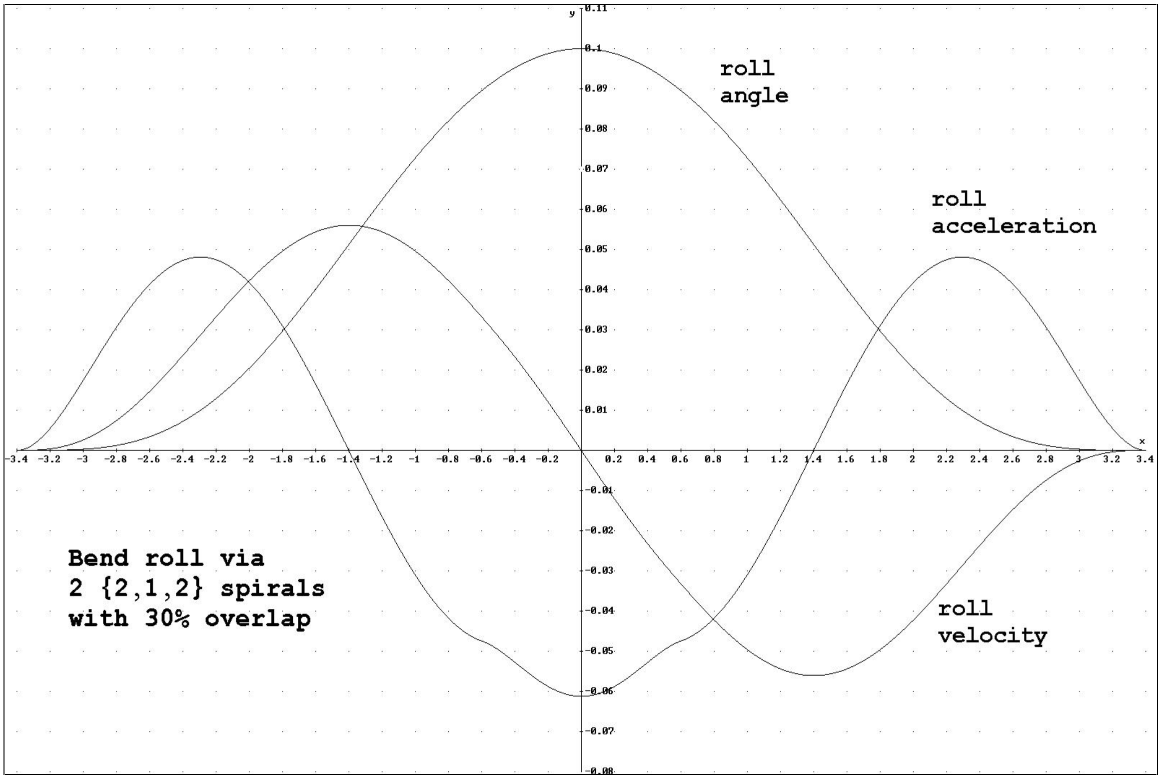

Figure 5. Roll functions for 2-{2,1} Bend with q = 0.7*a

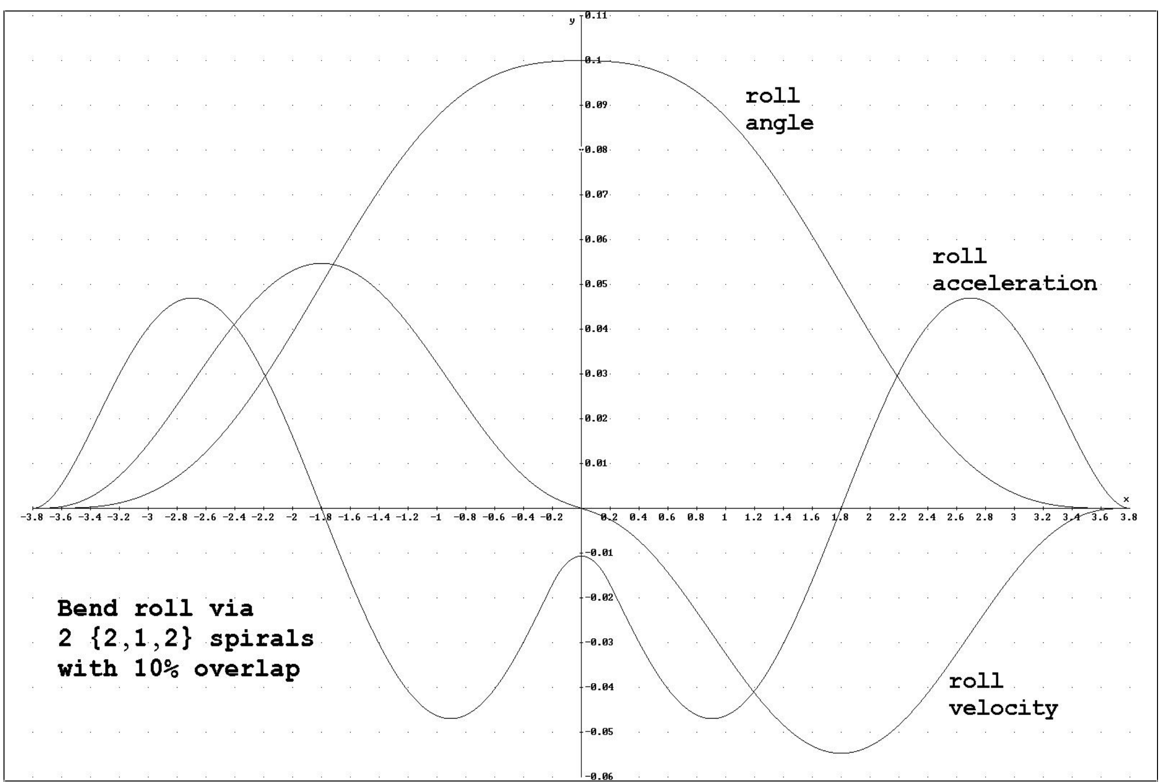

Figure 6. Roll functions for 2-{2,1} Bend with q = 0.9*a

Note with respect to equation (1) above that in railroad practice the roll angle will not normally be more than about 0.1 radians (6 inches elevation relative to a gage of about 60 inches) and that as a result the tangent of the roll angle will be nearly the same as the roll angle itself. Equation (1) thus indicates that the curvature at each point along the Bend will be approximately proportional to the roll angle at that point and hence that the total change of track bearing angle accomplished by a Bend will be approximately proportional to the integral of roll angle over the length of the Bend. Comparing the above three figures it can be observed that the area under the roll angle curve, and thus also the total turn angle of the Bend, decreases as q decreases. Note also that q cannot be lowered to 0, since with $q = 0$ the constituent roll accelerations would cancel and there would be no curvature at all.

Illustrations of track shapes derived from roll functions are provided for some of the roll functions that are defined below. However, the basic idea that applies in all cases can be seen via comparison of the plots of roll angle in Figure 2 and superelevation in Figure 3 with the plots of curvature and alignment in Figure 3.

The roll function family described above could be used in practice. There are some small unaesthetic inflections in the roll acceleration function in Figure 5 near $s = 0$, but they would have little effect on the dynamic performance of a corresponding Bend. However, we will present another approach that appears more attractive.

4. Constructing the roll acceleration for a Bend by inserting a factor.

In order to find another way to construct a roll acceleration function that will generate a Bend, we compare the simple spiral roll acceleration of Figure 2 with the Bend roll acceleration of Figure 4. In Figure 2 the roll acceleration crosses the s axis at $s = 0$ as a result of the presence of the factor s in equation (2). Looking at Figure 4 we observe that the roll acceleration function for a Bend crosses the s-axis once to the left of $s = 0$ and again to the right of $s = 0$. We can cause an s-axis crossing at $s = -f$ by inserting into equation (2) a factor of $(s + f)$, we can cause another s-axis crossing at $s = f$ by inserting a factor of $(s - f)$, and we can remove the crossing at $s = 0$ by dropping the factor of s. The resulting roll acceleration formula can be written as

$$\frac{d^2}{ds^2} r\_angle(s) = j \cdot (s+a)^2 \cdot (s-a)^2 \cdot (s+f) \cdot (s-f) \qquad (10)$$

where j is a multiplier to be determined. This roll acceleration extends from $s = -a$ to $s = a$ and is evidently symmetric about its mid point. It therefore qualifies as the roll acceleration for a Bend. We label this roll acceleration function in accordance with the orders of its zeros as {2,2} where the first 2 indicates that there is a second order zero at each end and the second 2 indicates that there are two first order zeros in the interior. Applying the constraint that the roll velocity must return to zero at the end of the Bend at $s = a$ we find that it is necessary to have $f = a / \sqrt{7}$, and allowing a redefinition of j the formula for the roll acceleration becomes

$$\frac{d^2}{ds^2} r\_angle(s) = j \cdot (a^2 - s^2)^2 \cdot (a^2 - 7s^2) \qquad (11)$$

The shapes of this roll acceleration function and of the corresponding roll velocity and roll angle functions are illustrated in Figure 7.

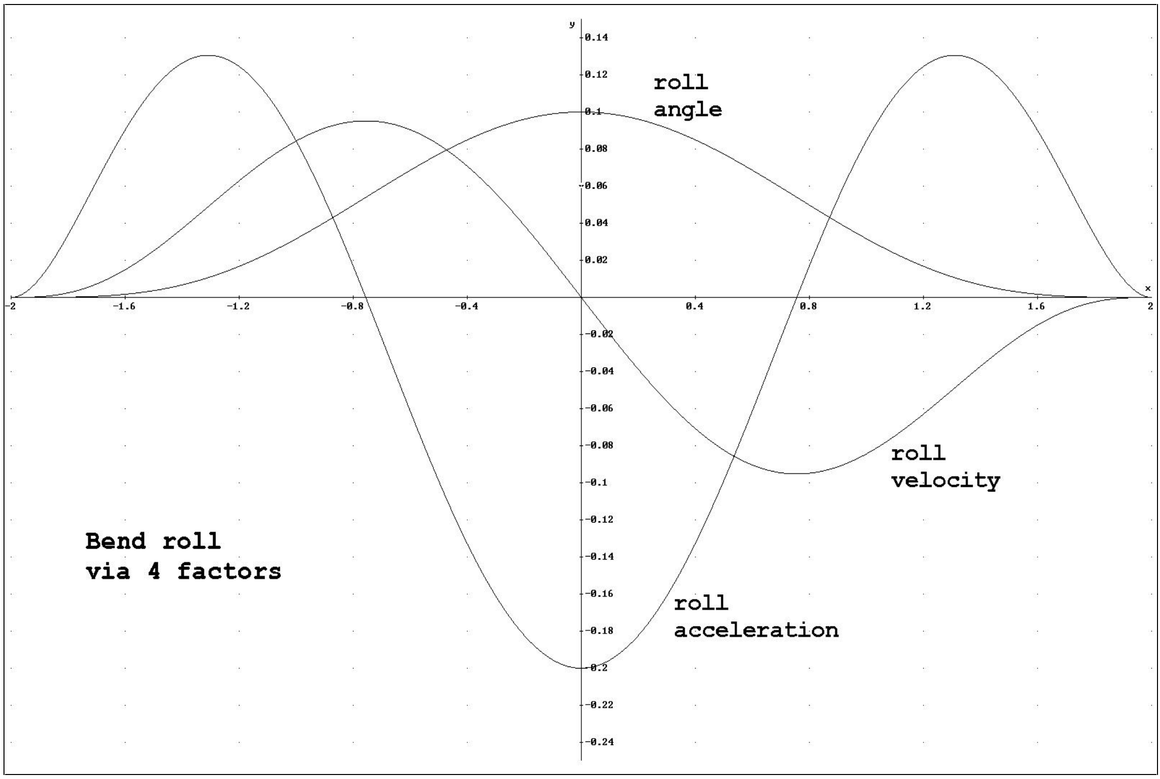

Figure 7. Roll functions for {2,2} Bend.

It may be observed that the roll functions of Figure 7 are similar in character to their counterparts in Figure 4.

Figures 8 and 9 show an example of a Bend alignment that was obtained by integrating equation (11) and that connects two adjacent sections of tangent track. In this example the angle of turn between the two tangents is 0.1 radians (= 5.73 degrees), the maximum superelevation of the track is 0.1 radians (= about 6 inches superelevation), the balancing speed of the Bend is set as 90 mph, and the height of the roll axis above the track is set at 8 feet..

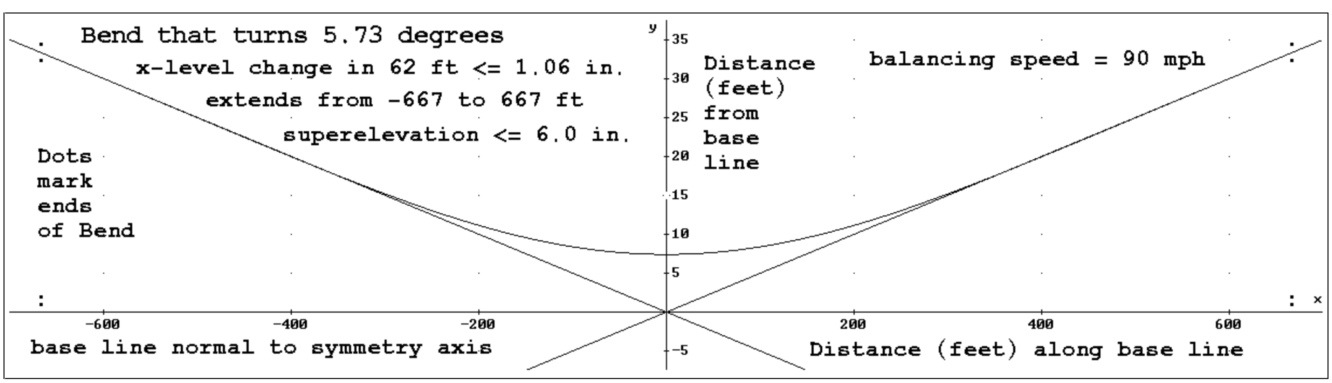

FIGURE 8. Alignment of a Bend connecting two tangents whose bearings differ by 0.1 radians (= 5.73 deg.) with track bank angle of 0.1 radians (about 6 inches superelevation) at the mid point, with length such that the balance speed is 90 mph, and with the track roll axis 8 feet above the track. The “y” axis is the symmetry line of the two tangents and the “x” axis is the base line that passes through the point of intersection of extensions of the tangents.

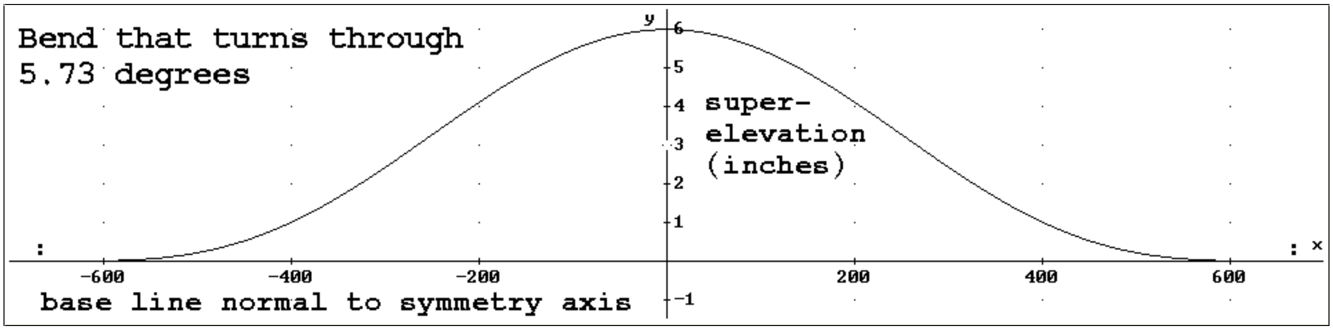

FIGURE 9. Track roll angle versus distance along the base line for the Bend of Figure 8.

For a Bend with parameters like these it is practical to introduce three approximations that simplify the mathematics. First, the tangent function in equation (1) is replaced by its argument (the bank angle in radians). Second and third, when integrating the sine and cosine of the track bearing angle, *b_axis(s)*, to obtain respectively the "y" and "x" coordinates of points on the path of the roll axis, the sine of the bearing angle is replaced by the bearing angle itself and the cosine of the bearing angle is replaced by unity. The effect of the second and third simplifications taken together is that *b_axis(s)* ceases to be the bearing angle and becomes in stead the tangent of the bearing angle. Therefore, when these simplifications are being applied *b_axis(s)* will be renamed as *bt_axis(s)* as a reminder. Alignments obtained based on these simplifications will differ from corresponding alignments obtained when the integrations are carried out numerically on the conceptually correct integrands. However, as long as roll angles and bearing angle changes do not exceed about 0.1 radians the differences of shape will be small and will not have adverse effects on the motions of vehicles traversing the Bends. The forgoing simplifications were used to obtain the alignment illustrated in Figure 8.

The algebra for this simplified application is as follows. The track roll angle as a function of distance obtained by integrating equation (11) twice can be written as

$$r\_angle(s) = k \cdot \left(a^2 - s^2\right)^4 \tag{12}$$

where k is a constant of convenience.

As a result of the third of the three approximations the integral for the "x" coordinate of a point on the path of the roll axis becomes trivial, and assuming the axes shown in Figure 8 we have the result $x = s$. This means that the parameter s no longer measures distance along the path of the roll axis and instead measures distance along the "x" axis. We therefore change the parameter in equation (1) from s to x. Continuing in the coordinate system illustrated in Figure 8, noting that the tangent of the bearing angle along the path of the roll axis will to be antisymmetric in x, we write

$$bt\_axis(x) = \frac{g}{v^2} \cdot \int_0^x dt \cdot r\_angle(t) \tag{13}$$

$$= \frac{g \cdot k \cdot x \cdot \left(315 \cdot a^8 - 420 \cdot a^6 \cdot x^2 + 378 \cdot a^4 \cdot x^4 - 180 \cdot a^2 \cdot x^6 + 35 \cdot x^8\right)}{315 \cdot v^2}$$

Pursuant to the second of the simplifications, the offset of the path of the roll axis from the base line (i.e., along the "y" axis in Figure 8) is given by the integral

$$y\_axis(x) = \int_{-a}^{x} dt \cdot bt\_axis(t) \tag{14}$$

$$= \frac{-g \cdot k \cdot \left(193 \cdot a^{10} - 315\, a^{8} \cdot s^{2} + 210\, a^{6} \cdot s^{4} - 126\, a^{4} \cdot s^{6} + 45 \cdot a^{2} \cdot s^{8} - 7 \cdot s^{10}\right)}{630\, v^{2}}$$

As already noted, with the path of the roll axis defined by equation (14) the compass bearing along it is given by

$$b\_axis(x) = \arctan\left(\frac{d}{dx} y\_axis(x)\right) = \arctan\left(bt\_axis(x) \right) \tag{15}$$

Returning to the formula for the displacement of the Bend from the base line, it may be observed that expression (14) is zero at each end of the Bend. To obtain the "y" coordinates of points on the path of the roll axis relative to the base line through the intersection of the two tangents as shown in Figure 8 it is necessary to add the "y" dimension from the base line to the points where the Bend meets the tangents, namely

$$\tan\left(\frac{turn}{2}\right) \cdot a \tag{16}$$

where $turn$ denotes the compass bearing of the second tangent relative to the first tangent. The track alignment is obtained from the path of the roll axis by subtracting the overhang illustrated in Figure 1, namely

$$o\_hang(x) = h \cdot \sin\left(r\_angle(x) \right) \tag{17}$$

where h represents the height of the roll axis above the track. Thus with the axes shown in Figure 8, the formula for the "y" coordinate of a point on the track becomes

$$y\_track(x) = y\_axis(x) + a \cdot \tan\left(\frac{turn}{2}\right) - h \cdot \sin\left(r\_angle(x)\right) \tag{18}$$

The primary constraint is that turn angle of the Bend be equal to $turn$. In light of equation (15) the equation that expresses that constraint is

$$bt\_axis(a) = \tan\left(\frac{turn}{2}\right) \tag{19}$$

and solving that equation for the multiplier k we obtain

$$k = \frac{315 \cdot \tan\left(\frac{turn}{2}\right) \cdot v^{2}}{128 \cdot a^{9} \cdot g} \tag{20}$$

There are two secondary constraints that place lower limits on the value of the half length a. One is that the roll angle of the track not exceed a maximum value denoted max_roll. That constraint is expressed by the equation $r\_angle(0) = \max\_roll$. Solving that equation for a provides a lower limit of

$$a\_roll\_\lim = \frac{315 \cdot v^2 \cdot \tan\left(\frac{turn}{2}\right)}{128 \cdot g \cdot \max\_roll} \tag{21}$$

The other secondary constraint is that the roll velocity along the track not exceed a value corresponding to the maximum allowed value of the twist of the track. That constraint is expressed by the equation

$$r\_veloc\left(\frac{-a}{\sqrt{7}}\right) = \max\_r\_veloc \tag{22}$$

where $r\_veloc(x)$ is the derivative of $r\_angle(x)$ with respect to x. Solving that equation for a the corresponding lower limit on the value of a is found to be

$$a\_twist\_\lim = \frac{9 \cdot (308700)^{1/4} \cdot v \cdot \sqrt{\tan\left(\frac{turn}{2}\right)}}{98 \cdot \sqrt{g} \cdot \sqrt{\max\_r\_veloc}} \tag{23}$$

The formulae for the minimum value of the half length, a, that follow from the secondary constraints on maximum roll angle and maximum roll velocity show simple dependence on the turn angle, *turn*, the balancing speed, v, and on the maximum roll angle or the maximum roll velocity. It is generally desirable to choose a value for a that is greater than both of the lower limits if the circumstances of the right of way so allow.

To obtain the distance along the track as a function of the "x" coordinate along the base line it is necessary to carry out a numerical integration. In light of equations (3) and (15) the formula is

$$s\_track(x) = \int_0^x dz \cdot \frac{1}{\cos\left\{\arctan\big(bt\_axis(z)\big) - \arctan\left[h\frac{d}{dz}\sin\big(r\_angle(z)\big)\right]\right\}} \tag{24}$$

Stepping back to compare the two forms of Bend at which we have looked, it can be noted that the roll acceleration formula in equation (11) is simpler than equation (9) that governs Figures 4, 5, and 6 partly because it does not have a parameter like the parameter q of equation (9) that provides an additional degree of freedom for constructing Bend shapes. Comparing the roll acceleration function of Figure 7 with that of Figure 6 we can observe that the turn angle of a Bend derived from the roll acceleration function of Figure 7 could be increased if we could find a convenient way to put a smooth dip in the magnitude of the roll acceleration near $s = 0$. We can do that by adding to equation (11) a factor of $(1 + q\, s^2)$ where q is an adjustable parameter. Again applying the constraint that the roll velocity should be zero at the end of the Bend and solving for the constant multiplier in terms of the maximum value of the roll angle we obtain the formula

$$d^2r/ds^2 = \frac{-\,120 \cdot \mathrm{max\_roll} \cdot (a + s)^2\,(a - s)^2\,(a^4\,q + 3 \cdot a^2\,(1 - q \cdot s^2) - 21 \cdot s^2) \cdot (q \cdot s^2 + 1)}{a^8 \cdot (a^4 \cdot q^2 + 22 \cdot a^2 \cdot q + 45)} \tag{25}$$

Setting the parameter q to zero causes equation (25) to become equivalent to equation (11). Increasing the parameter q from zero produces a family of Bends with increasing total turn angles. Figure 10 illustrates the roll motions obtained with $a = 2.0$, with the maximum roll angle set to 0.1 radians, and with q set to 5.0.

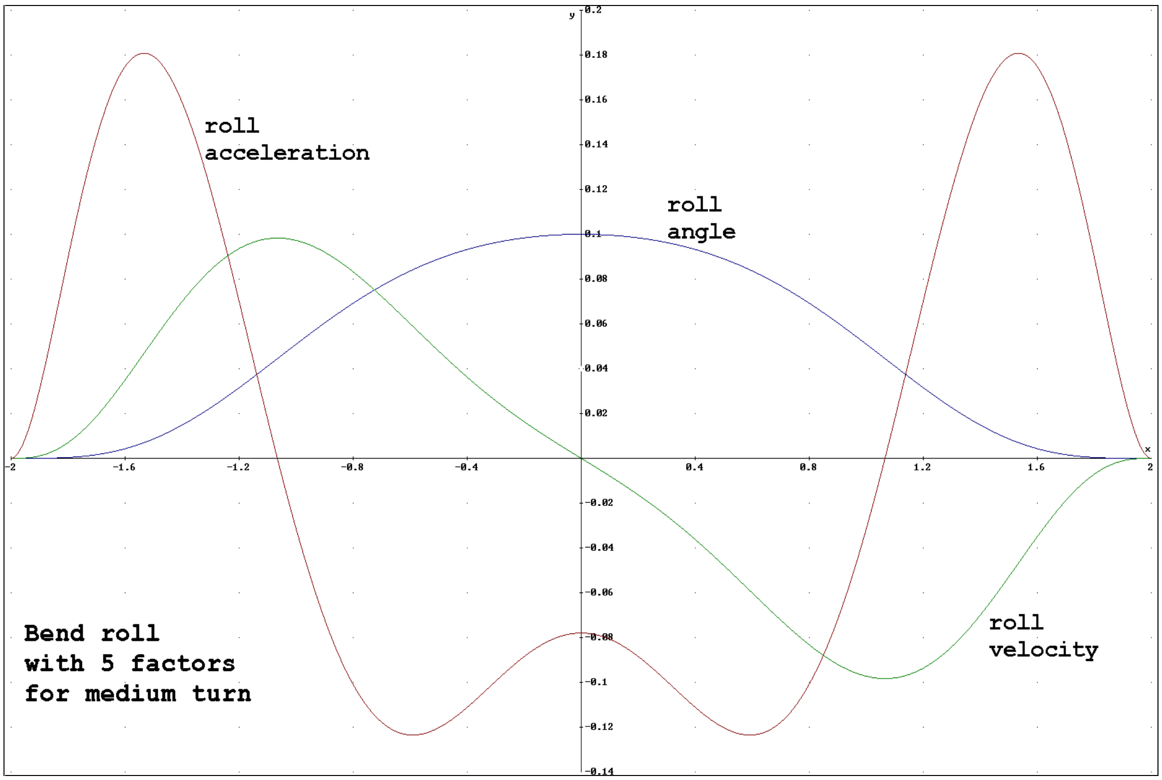

Figure 10. Roll functions for hybridized {2,2} Bend with q = 5.0.

While formula (25) could be used as the basis for a family of Bend shapes, it will be preferable in practice to use the more general approach that will be set forth in Section 9 below.

5. Constructing the roll acceleration for a Jog by overlapping three spirals

The term Jog as used herein refers to an alignment shape that begins tangent to one straight line, that moves smoothly away from that line toward a second straight line that is parallel to but not collinear with the first one, that ends tangent to the second line, and that is antisymmetric about its mid point. A crossover between adjacent parallel straight tracks provides an example of what a Jog looks like.

Recall that the roll acceleration for the 2-{2,1} Bend was formed by adding the roll acceleration of a spiral to the roll acceleration of another spiral with partial overlap. Analogously, by adding the roll acceleration of a {2,1} spiral to that of a 2-{2,1} Bend with the same partial overlap we obtain the roll acceleration of a 3-{2,1} Jog. The corresponding formula is

$$\begin{aligned} d^2r/ds^2 = & - \mathrm{BOX}(-a - q, s, a - q)\cdot j\cdot(s + q)\,(s + q - a)^2\,(s + q + a)^2 \\ & + 2\,\mathrm{BOX}(-a, s, a)\cdot j\cdot(s)\,(s - a)^2\,(s + a)^2 \\ & - \mathrm{BOX}(-a + q, s, a + q)\cdot j\cdot(s - q)\,(s - q - a)^2\,(s - q + a)^2 \end{aligned} \tag{26}$$

Because of the complexity of this formula it was not possible with the computing resources available during preparation of the present article to obtain a general formula for the values of s at which the magnitude of the roll associated with equation (26) has its maximum values. However, that would not prevent use of the formula in design work. A sense of what the roll functions look like can be gleaned from Figures 11 and 12 in which the functions are evaluated for $a = 2$, $j = 0.1$, and $q = 1.4$ (65% overlap) and 2.2 (45% overlap) respectively.

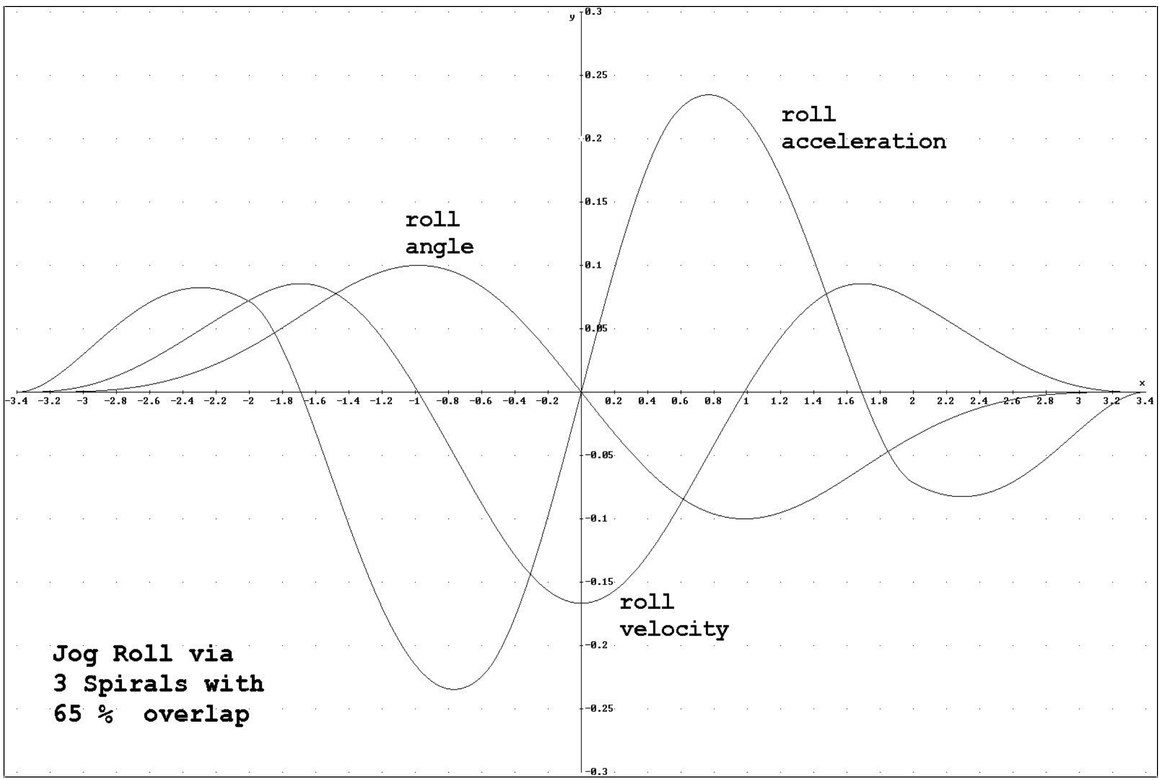

Figure 11. Roll functions for 3-{2,1} Jog with 65% overlap.

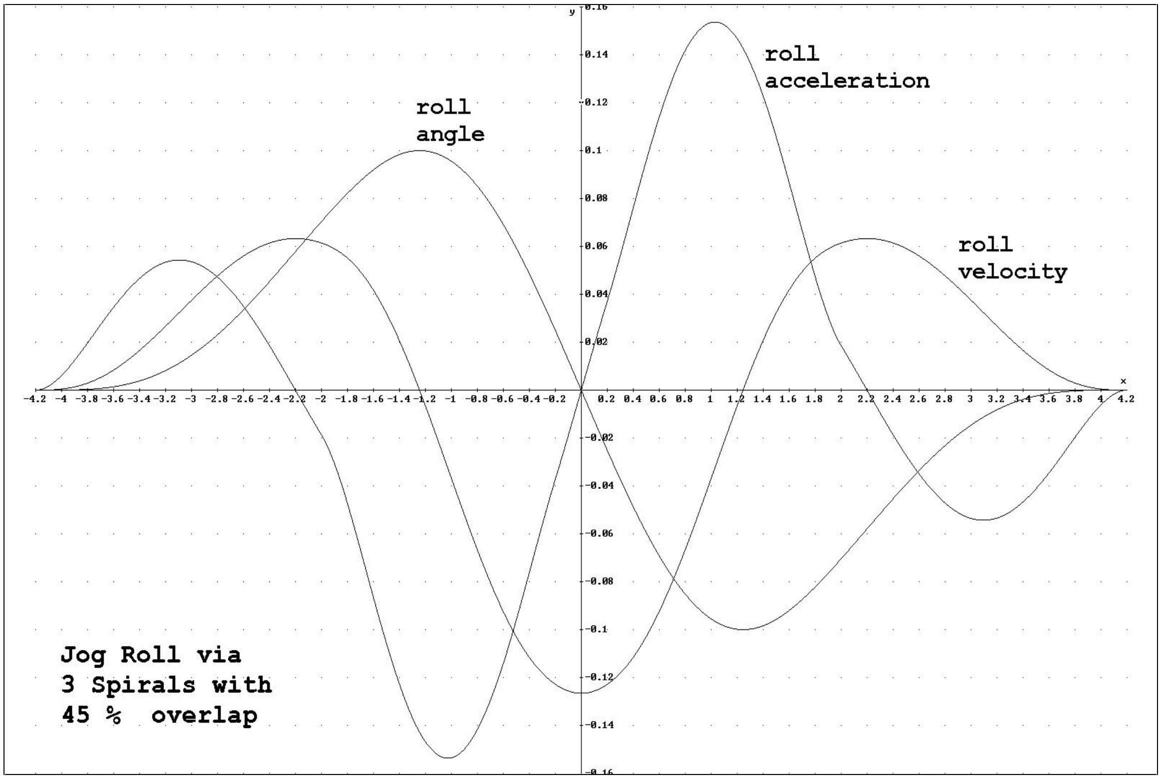

Figure 12. Roll functions for 3-{2,1} Jog with 45% overlap.

For overlap values below 45% or above 65% the roll acceleration curves become less smooth and hence presumably less desirable for design of Jogs for railroad track. Even with the parameters of Figures 11 and 12 that are usable, one can observe unaesthetic inflections in the acceleration curves. The combination of that lack of smoothness and the mathematical complexity of the algebra of equations like equation (26) make Jogs of the above type unappealing in comparison to those that are presented in the following Sections.

6. Constructing the roll acceleration for a Jog by inserting a factor.

We now look at a second way to form the roll acceleration function for a Jog. Just as each of the Bend formulae has one more root (or s-axis crossing) than the corresponding Spiral formula, so, each of the three curves (roll acceleration, roll velocity, and roll angle) defining the roll motion of a Jog needs to have one more root than the corresponding curve for a Bend. Starting from equation (10) which was the initial equation for a Bend and which is symmetric about the mid point, we can both add another root and make the resulting function antisymmetric by inserting a factor of s so that the added root is at the mid point of the function. The result is

$$\frac{d^2}{ds^2} r\_angle(s) = j \cdot (s+a)^2 \cdot (s+f) \cdot s \cdot (s-f) \cdot (s-a)^2 \qquad (27)$$

where f *is* between 0 and a. Integrating that equation twice to obtain the formula for the change in roll angle over the length of the Jog and requiring that the change in roll angle be zero at the end of the Jog, we find that we must set $f = a/\sqrt{3}$. If we then solve for the constant multiplier in terms of the maximum roll angle in the Jog, *max_roll*, the formula for the multiplicative Jog becomes

$$\frac{d^2}{ds^2} r\_angle(s) = \frac{-\ 59049 \cdot \mathrm{max\_roll} \cdot s \cdot (a^2 - s^2)^2 \, (a^2 - 3s^2)}{512 a^9} \qquad (28)$$

Figure 13 illustrates the shapes of the roll functions for the multiplicative Jog.

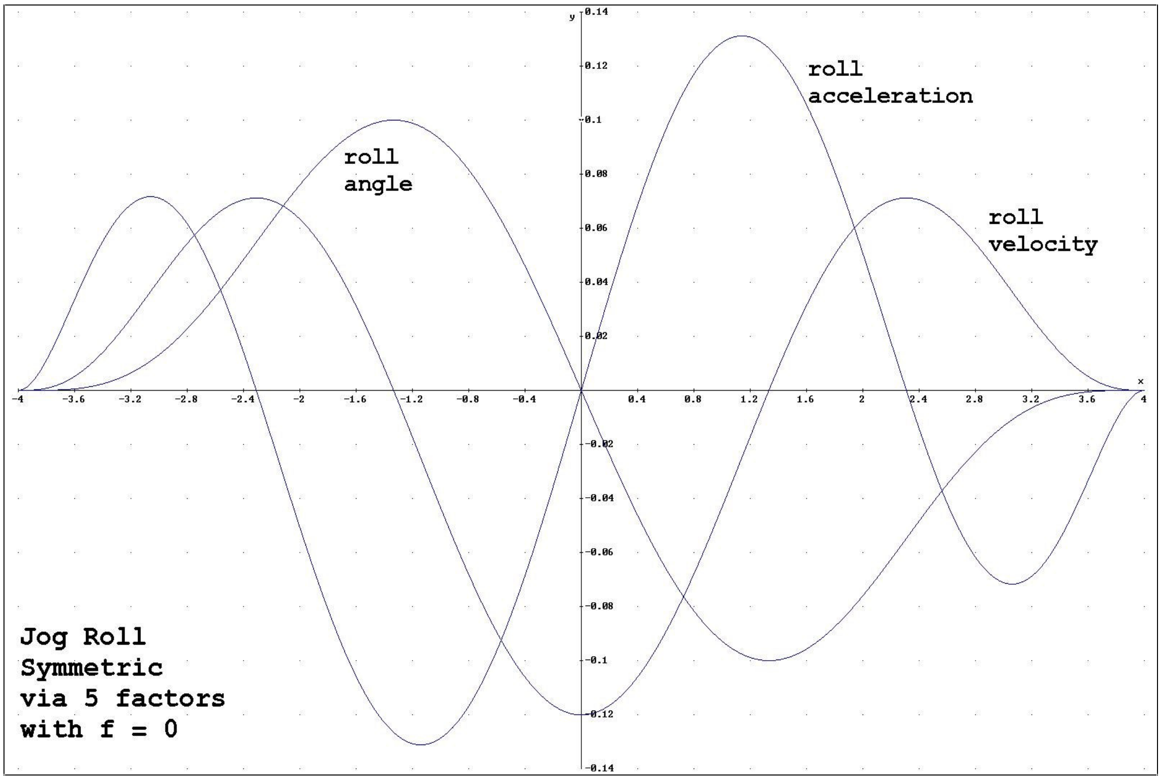

Figure 13. Symmetric (i.e., antisymmetric) Jog roll functions via 5 factor roll acceleration.

The overall turn angle of the antisymmetric Jog can be seen to be zero because the curve for the roll angle is antisymmetric about the mid point of the Jog.

7. Jogs as Shapes for Turnouts and Crossovers

This Section configures a jog intended to serve as the alignment for a crossover between two parallel tangent (i.e., straight) tracks. As the track bearing angles relative to the two tangents and the track roll angles are small, it is reasonable to use the simplified treatment set forth in Section 4 above.

The formula for the roll angle of the symmetric Jog as a function of length along the track is obtained by integrating equation (28) twice and can be written in the form

$$r\_angle(s) = -k \cdot s \cdot \left(a^2 - s^2\right)^4 \qquad (29)$$

where k is a constant of convenience. The corresponding form of equation (28) for the roll acceleration is

$$r\_accel(s) = -24 \cdot k \cdot s \cdot \left(a^2 - s^2\right)^2 \left(3 \cdot s^2 - a^2\right) \tag{30}$$

The minus sign is inserted so that for k positive, at the left end of the Jog where $s < 0$, the initial bank and turn will be positive which is interpreted as being to the left.

Applying the first small angle approximation, the integral for the tangent of the bearing angle along the path of the roll axis versus distance, x, along the tangents becomes

$$bt\_axis(x) = \frac{g}{v^2} \cdot \int_{-a}^{x} dt \cdot r\_angle(t) = \frac{\mathrm{g} \cdot \mathrm{k} \cdot \left(\mathrm{a}^2 - \mathrm{x}^2\right)^5}{10 \cdot \mathrm{v}^2} \tag{31}$$

Applying the second small angle approximation, the integral for the coordinate of a point on the roll axis along a “y” axis normal to the two tangents and with zero value mid way between them is

$$y\_axis(x) = \int_{0}^{x} dt \cdot bt\_axis(t) \tag{32}$$

$$= \frac{g \cdot k \cdot x \,(\, 693\ a^{10} - 1155\ a^8\ x^2 + 1386\ a^6\ x^4 - 990\ a^4\ x^6 + 385\ a^2\ x^8 - 63\ x^{10}\,)}{6930\ v^2}$$

In this situation the primary constraint is that the lateral displacement over the length of the Jog, denoted *jog_dist*, should equal the specified distance between the centerlines of the two parallel tangent tracks. With the roll angle zero at each end of the Jog there is no overhang at either end, and this constraint takes the form

$$y\_axis(a) = \frac{jog\_dist}{2} \tag{33}$$

Solving that constraint for k we find

$$k = \frac{3465 \cdot jog\_dist \cdot v^2}{256 \cdot g \cdot a^{11}} \tag{34}$$

The secondary constraints are that the roll angle and twist of the track should nowhere exceed the respective limits chosen for those two properties. (The roll velocity determines the track twist.)

The maximum value of roll angle occurs at $s = -a/3$ and $s = a/3$ so that this constraint is expressed as

$$r\_angle(-a/3) = \max\_roll \tag{35}$$

and the value of shape half length such that the maximum magnitude of the roll angle is *max_roll* is found to be given by

$$a\_roll\_\lim = \frac{4 \cdot \sqrt{1155} \cdot \sqrt{jog\_dist} \cdot v}{81 \cdot \sqrt{g} \cdot \sqrt{\max\_roll}} \tag{36}$$

The magnitude of the roll velocity has its maximum value at $s = 0$ and the value of the shape half length such that the magnitude of the roll velocity there equals *max_r_veloc* is found to be given by

$$a\_twist\_\lim = \frac{{}^{1/3}\sqrt[3]{6930} \cdot jog\_dist^{1/3} \cdot v^{2/3}}{8 \cdot g^{1/3} \cdot \max\_r\_veloc^{1/3}} \tag{37}$$

Both of the above expressions for the half length are evaluated, and the larger value is used so that both of the secondary constraints are satisfied.

Both formulae for the Jog's half length show that the length of the Jog will depend in a simple way on the balancing speed, v, the Jog's lateral displacement, *jog_dist*, and either the maximum bank angle, *max_roll*, or the maximum roll velocity, *max_r_veloc*.

Since we are looking here at crossovers we need to take account of the fact that physical crossovers begin and end with physical track switches. Physical track and guide way switch design is an immense field in which many concepts for balancing cost and performance have been developed. What is noted here is that costs of constructing and maintaining a switch are increased when there is an increase in the length of the assembly that must move when the setting of the switch is changed. That length increases when there is an increase in the track length over which geometry prevents a guide rail or rails from simultaneously being in the working location for the both routes. This means that it is desirable to arrange to have the initial lateral separation of the "diverging" path from the "through" path develop as rapidly as vehicle dynamics will allow. We have observed that raising the height of the vehicle roll axis above the track is always dynamically beneficial. However, raising the vehicle roll axis also causes an increase in the distance from the start of a Jog to the point at which the Jog reaches a given lateral displacement toward the final tangent. Therefore, in contrast to the application of the Bend in Section 4 above, for this application of a Jog as a crossover the roll axis is not raised and the path of the track is given by equation (32) itself. We thereby make some sacrifice of dynamic performance in order to reduce the cost of the crossover.

Figure 14 illustrates such a crossover for the conditions that the two adjacent sections of tangent track have a centerline separation of 20 feet, that the maximum superelevation in the crossover is 0.05 radians (about 3 inches superelevation), and that the balance speed of the crossover is 90 mph. The crossover extends for 781 feet in each direction from the center of symmetry. Figure 16 shows the track roll angle profile corresponding to Figure 14 as given by equation (30). (Formulae for Figures 15 and 17 are presented following Figure 17.)

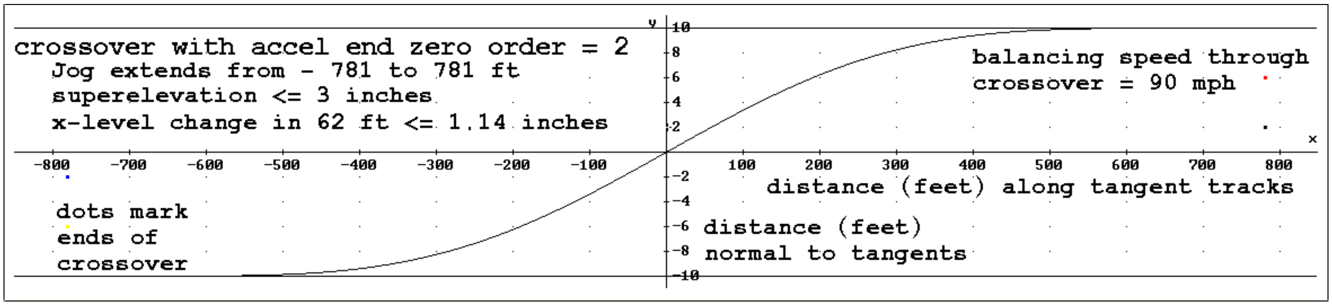

Figure 14. A Jog constructed from the roll acceleration of equation (30) which has a 2nd order zero at each end. This Jog is configured as a crossover between tangent tracks with centerline spacing of 20 ft, with maximum roll angle of 0.05 radians (corresponding to maximum superelevation of about 3 inches), and with dimensions chosen so that the balancing speed is 90 mph. The total length of the crossover measured along the tangents is 1,562 ft. (Note that in North American practice crossovers are not usually designed for high speed operation and do not usually include superelevation.)

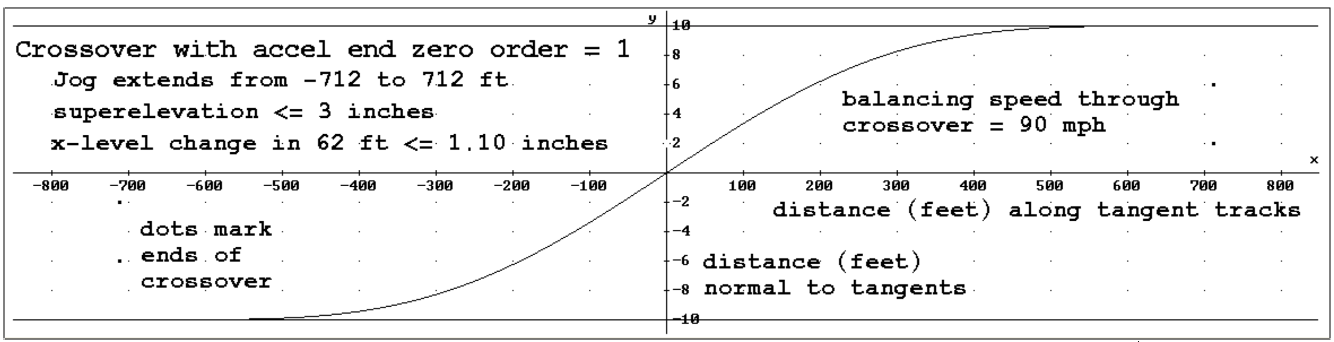

FIGURE 15. A Jog constructed from the roll acceleration of equation (30) with a 1st order zero at each end. Other design parameters are the same as those for Figure 14. Because the roll angle builds more quickly at each end than is the case when the roll acceleration has 2nd order zeros at the ends, this crossover is shorter than the one in Figure 14. The total length of this crossover measured along the tangents is 1,424 ft.

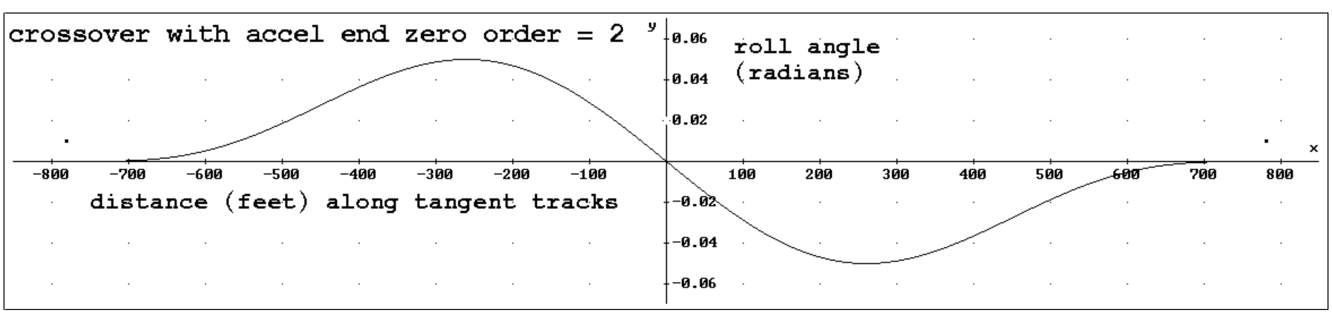

FIGURE 16. Track roll angle for Jog of Figure 14.

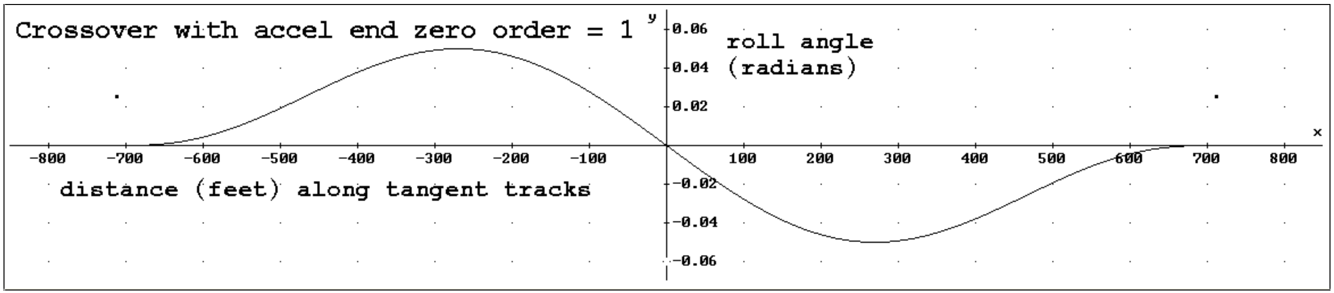

FIGURE 17. Track roll angle for Jog of Figure 15.

We look next at the rationale of Figures 15 and 17. As noted at the end of Section 2 above, when the roll axis is not raised above the plane of the track it is reasonable to look at roll acceleration functions that have 1st rather than 2nd order zeros at the ends. Starting with the counterpart of equation (27) but with 1st order rather than 2nd order zeros at $s = -a$ and $s = a$ and repeating the sequence of steps that lead from equation (27) to equation (28), one finds that the counterpart of equation (28) with a 1st order zero at each end is

$$\frac{d^2}{ds^2} r\_angle(s) = \frac{343\sqrt{7} \cdot \max\_roll\ (a+s)(a-s)(3a^2 - 7s^2)}{36a^7} \tag{38}$$

The formula for the roll angle of the symmetric Jog obtained by integrating equation (38) twice can be written in the form

$$r\_angle(s) = -k\,s\,(a^2 - s^2)^3 \tag{39}$$

where k is another constant of convenience.

Applying the simplified treatment as in the previous case the integral for the tangent of the bearing angle versus distance, x, along the tangents becomes

$$bt\_axis(x) = \frac{g}{v^2} \cdot \int_{-a}^{x} dt \cdot r\_angle(t) = \frac{\mathrm{g} \cdot \mathrm{k}\ (\mathrm{a}^2 - \mathrm{x}^2)^4}{8\,\mathrm{v}^2} \tag{40}$$

Applying the second small angle approximation, the integral for the distance along a “y” axis normal to the two tangents is

$$y\_axis(x) = \int_0^x dt \cdot bt\_axis(t) \tag{41}$$

$$= \frac{g \cdot k \cdot x\ (315\ a^8 - 420\ a^6\ x^2 + 378\ a^4\ x^4 - 180\ a^2\ x^6 + 35\ x^8)}{2520\,v^2}$$

The primary and secondary constraints and the manner in which they are used to determine k and a are the same as in the previous case. The solution for k is

$$k = \frac{315 \cdot jog\_dist \cdot v^2}{32 \cdot a^9\ g} \tag{42}$$

The lower limit for a such that the *max_roll* constraint is satisfied is

$$a\_roll\_\lim = \frac{9 \cdot (77175)^{1/4} \cdot \sqrt{jog\_dist} \cdot v}{98 \cdot \sqrt{g} \cdot \sqrt{\max\_roll}} \tag{43}$$

and the lower limit for a such that the *max_r_veloc* constraint is satisfied is

$$a\_twist\_\lim = \frac{\sqrt[1/3]{630} \cdot jog\_dist^{1/3} \cdot v^{2/3}}{4 \cdot g^{1/3} \cdot \max\_r\_veloc^{1/3}} \tag{44}$$

The constraints on the Jog's half length show the same dependence as before on the balancing speed, the Jog's lateral displacement, and on the maximum bank angle or maximum roll velocity but with different constant factors.

Figure 15 illustrates a crossover based on the above formulae for the conditions that the two adjacent sections of tangent track have a centerline separation of 20 feet, that the maximum superelevation in the crossover is 0.05 radians (about 3 inches superelevation), and that the balance speed of the crossover is 90 mph. The crossover extends for 712 feet in each direction from the center of symmetry. Figure 17 shows the track roll angle profile corresponding to Figure 15 as given by equation (39).

In contemporary North American railroad practice turn-outs from tangent tracks do not incorporate superelevation and therefore do not have defined balancing speeds. Construction of a switch that incorporated superelevation as prescribed by the formulae of this Section would require progressive lowering of the rail seats of the low rail of the diverging route and would require a novel machining of the "point" for the through route. The points would also be longer than the points of conventional railroad switches.

8. Constructing the roll acceleration for a Wiggle by inserting a factor

The term Wiggle as used herein refers to an alignment shape that begins tangent to some straight line, that makes a smooth lateral excursion away from and then back toward that straight line, that ends again tangent to that straight line, and that is symmetric about its mid point. (We will see in Section 9 below that asymmetry can be accommodated by addition of higher order shapes.) As noted in the introduction, a Wiggle can be an effective way for an otherwise straight alignment to circumvent a local obstruction that intrudes from one side. We construct a roll acceleration formula that can exhibit the features of a Wiggle by adding another linear factor to the Jog roll acceleration formula in equation (27). The initial formula is

$$\frac{d^2}{ds^2} r\_angle(s) = j(s+a)^2(s+p)(s+f)(s-q)(s-i)(s-a)^2 \tag{45}$$

The parameters p and i are eliminated by applying the constraints that the roll velocity and roll angle both return to zero at $s = a$. The asymmetry of the resulting roll acceleration polynomial is controlled by $(f - q)$. It is therefore convenient to replace f and q by the new variables $b = (f +$

$q)/2$ and $c = (f - q)/2$. We want these shapes to be symmetric so that c is set to zero and drops out. This makes the roll acceleration a polynomial in s that depends on j, a, and b.

The following figures illustrate the shape of the roll angle function for two values of the parameter b.

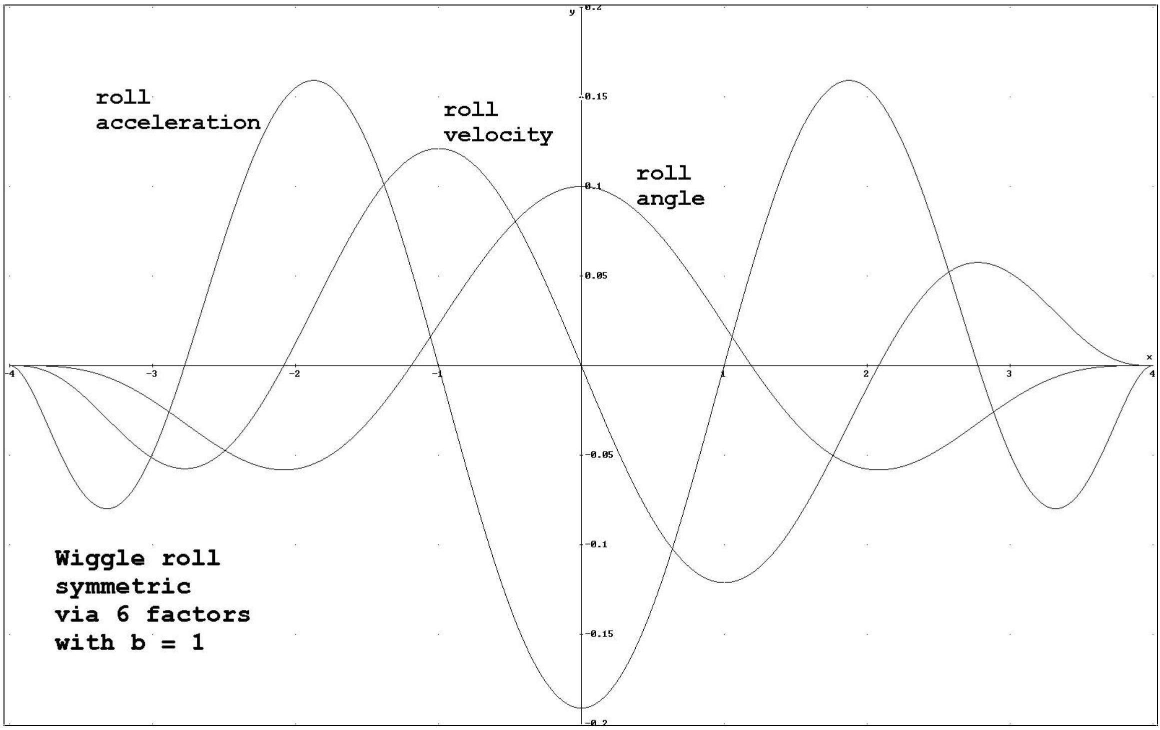

Figure 18. Symmetric roll functions via 6 factor roll acceleration with b = 1.0.

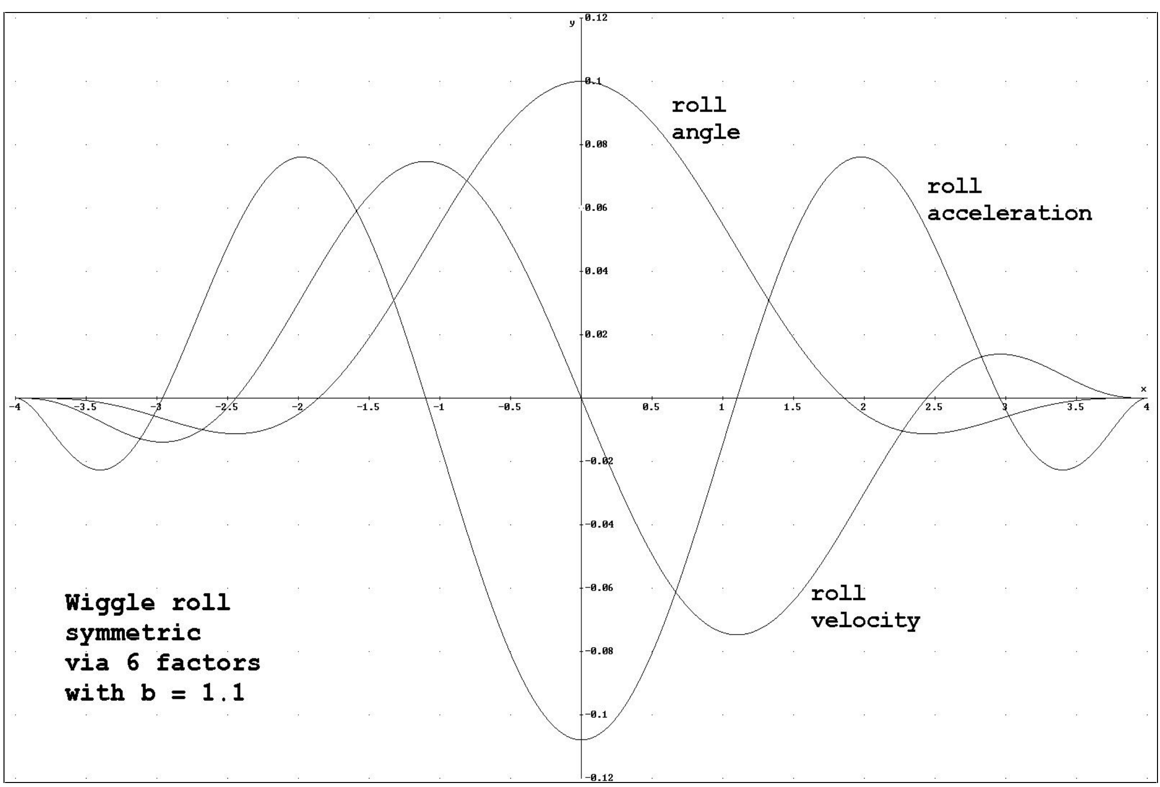

Figure 19. Symmetric roll functions via 6 factor roll acceleration with b = 1.1.

Looking in Figure 18 at the areas between the roll angle curve and the distance axis one can see that the amounts of curvature to the right and to the left are approximately equal so that the total angle of turn over the length of the shape will approximate the desired value of zero. By way of contrast, the roll angle curve of Figure 19 is biased to one side so that the resulting shape will look much like a Bend and not much like a Wiggle.

The roll angle function corresponding to equation (45) is a 10^{th} order polynomial. To obtain a closed form algebraic expression for the constraint that the compass bearing of the Wiggle be the same at the end as at the beginning would require putting that 10^{th} order polynomial roll angle function into equation (1) and then obtaining the compass bearing angle as the integral of equation (1) in closed form. As that is considered impossible we will provide an illustration using the simplified method described in Section 4 above. The formula for the tangent of the bearing angle on the path of the roll axis is then available in closed form. Imposing the requirement that the tangent of the bearing angle be the same at the end as at the beginning fixes the value of b, and in the context of the simplified treatment the equation for the roll acceleration of a Wiggle (with j redefined) becomes

$$\frac{d^2}{ds^2} r\_angle(s) = j(a^2 - s^2)^2 (a^4 - 18a^2 s^2 + 33 s^4) \qquad (46)$$

What comes next is an example of application of a Wiggle to avoid a single obstacle in an otherwise straight section of track. The illustration makes use of the mathematical simplifications that were explained in Section 4 above. The roll angle function corresponding to equation (46) can be written as

$$r\_angle(s) = k\,(a^2 - s^2)^4\,(11\,s^2 - a^2) \qquad (47)$$

where k is a constant of convenience. Integrating the simplified version of equation (1) yields

$$bt\_axis(x) = \frac{-g \cdot k \cdot x \cdot (a^2 - x^2)^5}{v^2} \qquad (48)$$

and evaluating the simplified form of the integral for *y_axis* yields

$$y\_axis(x) = \frac{g \cdot k \cdot (a^2 - x^2)^6}{12\, v^2} \qquad (49)$$

The “y” coordinate along the path of the track is

$$y\_track(x) = y\_axis(x) - \cos\left(\arctan\left(bt\_axis(x)\right)\right) \cdot o\_hang(x) \qquad (51)$$

where *o_hang(x)* is the overhang as described previously.

The primary constraint is that the lateral excursion from the general tangent have a specified value that we denote by *swing_dist* and takes the form

$$y\_track(0) = swing\_dist \tag{52}$$

It is evident from equation (48) that $bt\_axis(0) = 0$. *Therefore* when equation (51) is used in equation (52) the cosine factor is unity and can be dropped. To get the constraint into a form that can be solved algebraically we can introduce another approximation that is in keeping with the simplified treatment. Namely, in the equation for the overhang we replace the sine of the roll angle by the roll angle itself and write

$$o\_hang(x) = h \cdot r\_angle(x) \tag{50}$$

We can then solve for k and find

$$k = \frac{12 \cdot swing\_dist \cdot v^2}{a^{10}(a^2\, g + 12 \cdot h \cdot v^2)} \tag{53}$$

There are two secondary constraints. One is that the maximum roll angle that occurs at $s = 0$ not exceed a specified value we denote as *max_roll*. The lower limit for a obtained by solving that constraint is

$$a\_roll\_\lim = \frac{2 \cdot \sqrt{3} \cdot v \cdot \sqrt{(h \cdot \max\_roll + swing\_dist)}}{\sqrt{g} \cdot \sqrt{\max\_roll}} \tag{54}$$

The other secondary constrain is that the maximum roll velocity (which corresponds to maximum allowed track twist) which occurs at $-a \cdot \sqrt{\left(3/11 - 4\sqrt{3}/33\right)}$ not exceed a value we denote as *max_r_veloc*. The lower limit for a obtained by solving that constraint is

$$a\_twist\_\lim = -4 \cdot \sqrt{-1} \cdot v\sqrt{\frac{h}{g}} \cdot \sin\left\{\frac{theta}{3}\right\} \tag{55}$$

where

$$theta = \arcsin\left\{\frac{\sqrt{-1} \cdot \sqrt{(h \cdot g)} \cdot swing\_dist \cdot \sqrt{\left(\frac{1517158400 \cdot \sqrt{3}}{526153617} + \frac{454246400}{58461513}\right)}}{h^2 \cdot \max\_r\_veloc \cdot v}\right\} \tag{56}$$

The values used for the example illustrated in Figures 20 and 21 are: swing distance = 20.0 feet; balancing speed = 90 mph; roll axis height = 8.0 feet; maximum roll velocity corresponding to a maximum change of cross level in 62 feet = 1.2 inches; and the acceleration of gravity = 32.17 feet/second_squared. With the selected parameters the roll angle does not get as large as the typical limit of 0.1 radians. While the alignment given by this simplified construction is not identical to the alignment that would be obtained if all of the trigonometric functions of the method were fully evaluated, its utility and dynamic characteristics will be just as good as those of a corresponding construction with the trigonometric functions fully evaluated. However, if this simplified treatment is used in practice, the engineer will need to be aware that the relationship between curvature and superelevation is slightly different than normal and will need to take account of that when establishing authorized speeds.

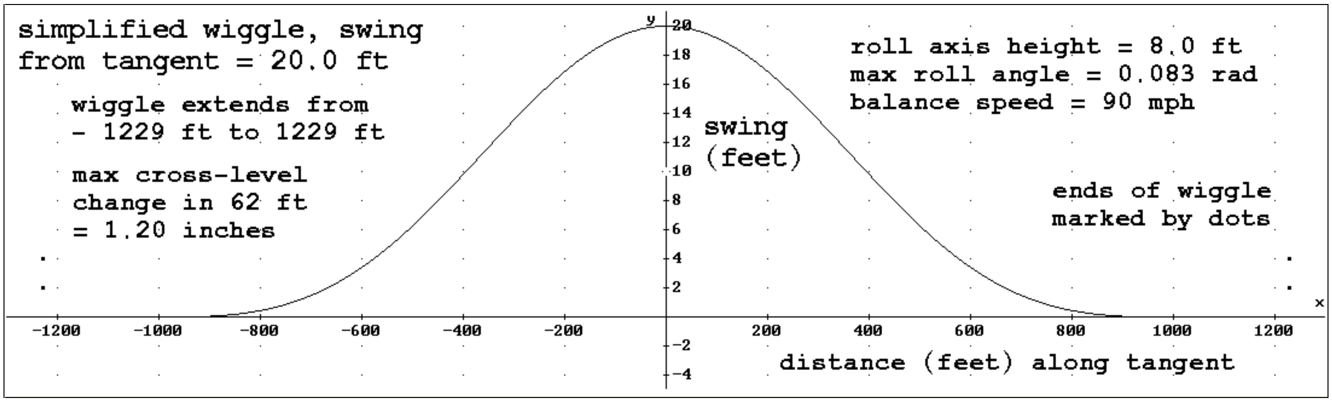

FIGURE 20. Illustration of a simplified Wiggle that swings 20.0 feet laterally to avoid an obstacle along what is otherwise straight track.

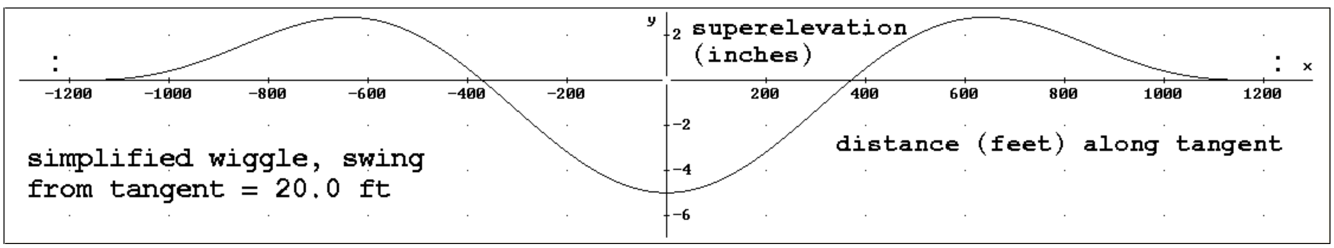

FIGURE 21. The superelevation (assuming a 60.0 inch gage) of the same Wiggle as in Figure 20.

9. Simplification and Generalization by means of Gegenbauer Polynomials

We next look again at equation (11)

$$\frac{d^2}{ds^2} r\_angle_{bend}(s) = j \cdot (a^2 - s^2)^2 \cdot (a^2 - 7s^2) \tag{11}$$

that gives the roll acceleration function for a Bend. That equation is integrated once to obtain the roll velocity function and a second time to obtain the roll angle function. The fact that we need to work with the integral of $(a^2 - s^2)^2$ times a polynomial suggests a check to see if one of the classical orthogonal polynomial families has a weighting function that can take the form $(a^2 - s^2)^2$. Consulting a treatise on orthogonal polynomials such as Reference 6, one finds that the Gegenbauer polynomials denoted $C_n^{(\alpha)}(x)$ are defined with respect to the weighting function $(1 - x^2)^{(\alpha - ½)}$ which with suitable choices for the variables can take the form with which we are dealing. Starting with equation 22.13.2 of Reference 6, defining $m = \alpha - ½$, scaling the variable of integration so that the limits of integration are from $-a$ to a, and working with $n \geq 1$, one can obtain the relation

$$\int_{-a}^{s} dt \cdot \left(a^2 - t^2\right)^m C_n^{(m+1/2)}(t/a) = \frac{-(2m+1)}{n(2m+1+n)} \frac{\left(a^2 - s^2\right)^{(m+1)}}{a} C_{n-1}^{(m+3/2)}(s/a) \tag{57}$$

It may be noted that $C_n^{(\alpha)}(x)$ is a polynomial of order n that is an even or odd function of x depending on whether n is an even or odd integer.

Looking at the explicit expression for $C_2^{(5/2)}(s/a)$ and introducing a constant j_2 it is easy to verify that equation (11) that gives the roll acceleration for a Bend can be rewrite as

$$\frac{d^2}{ds^2} r\_angle_{bend}(s) = j_2 \cdot \left(a^2 - s^2\right)^2 C_2^{(5/2)}(s/a) \tag{58}$$

In the same way one can verify that the right sides of equations (2), (28), and (46) that give the roll acceleration functions for a Spiral, for a Jog, and for a Wiggle can be written as

$$j_n \cdot \left(a^2 - s^2\right)^m C_n^{(m+1/2)}(s/a) \quad , \quad n \geq 1, \tag{59}$$

with $n = 1, 3,$ and 4 respectively and with j_n denoting the weighting coefficient for the order n contribution. We are therefore lead to look upon a series of terms of the form of expression (59) with $n >= 1$ as the roll acceleration of defining a general connecting shape in which the curvature varies with distance over the length of the shape.

By virtue of equation (57) the equation for roll velocity obtained by integrating expression (59) once will contain a factor of $(a^2 - s^2)^3$. For terms with $n >= 2$ the contributions to the roll angle obtained by integrating a second time will contain a factor of $(a^2 - s^2)^4$. Thus the two constraints that the roll velocity and roll angle both return to zero at $s = a$ will be satisfied automatically for terms with $n >= 2$.
Applying equation (57) twice the contribution of a term with $n >= 2$ to the roll angle is found to be

$$\frac{j_n\,(2m+1)\,(2m+3)}{n\,(n-1)\,(2m+1+n)\,(2m+2+n)}\;\frac{(a^2-s^2)^{(m+2)}}{a^2}\,C_{n-2}^{(m+5/2)}(s/a) \tag{60}$$

It was pointed out in Section 8 that with the Wiggle roll acceleration given by equation (46) or equivalently by $j_4 \cdot \left(a^2 - s^2\right)^2 C_4^{(5/2)}(s/a)$ the total compass bearing change over the length of the Wiggle will vanish as it should if the "simplified" treatment is used but not if the trigonometric integrands are fully evaluated. Therefore, in a "full" treatment that includes even order Gegenbauer terms with $n >= 4$ it is necessary to include at least a little $n = 2$ or Bend component and to adjust the strength of that component so that the net change of compass bearing over the length of the shape has the required value.

As a rule-of-thumb, the "simplified" method can be used with no problem for a shapes that connects to a tangent at each end, and the "full" method should be used for any shape that connects at one or both ends to a curved arc.

It should be noted that equation (57) holds for non-integral values of m provided that $m > -1$. As an example, choosing $m = 2.5$ would cause the roll acceleration to rise from zero more slowly at each end of the shape but would then mean that roll acceleration values would be greater in the interior of the distance covered by the shape. Choosing $m = 1.5$ would have the opposite effects.

Finally we observe that if the roll acceleration includes a $j_n\left(a^2-s^2\right)^m C_n^{(m+1/2)}(s/a)$ term with $n=1$ then the second integration to obtain the contribution to the roll angle requires special treatment. In this case the contribution to roll angle is

$$\frac{-j_1(2m+1)}{a\cdot(2m+2)}\int_{-a}^{s} dt\cdot\left(a^2-t^2\right)^{m+1}. \tag{61}$$

When the above integral is evaluated for $s=a$ and plotted as a function of m it is found that that for $m > 0$ the integral itself is positive and is approximately proportional to $e^{(8m/3)}$.

For $m=2$ expression (61) becomes

$$-j_1\left(16\,a^7+35\,a^6 s-35\,a^4 s^3+21\,a^2 s^5-5\,s^7\right)/(42\,a) \tag{62}$$

and this expression is the roll function of the order {2,1} improved spiral described in Section 2 above.

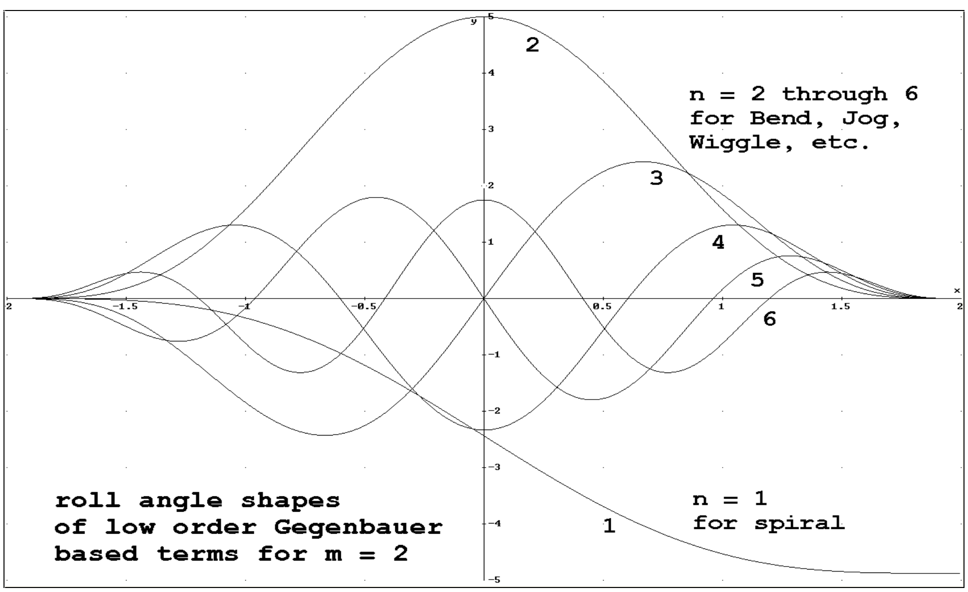

Figure 22. Shapes of contributions by expressions (62) and (60) to the track roll angle when m = 2.

Figure 22 illustrates the shapes of contributions to the roll angle by expressions (62) and (60) when $m=2$ with the respective j_n factors all equal to unity.

When there is need for a shape that includes some bending, jogging, and/or wiggling and in addition connects adjacent arcs whose curvatures differ, then the combination of Gegenbauer based terms will need to include an $n=1$ contribution.

Summarizing, we have found that a track or guide way shape that incorporates a combination of spiral, Bend, Jog, and Wiggle features can be obtained from a roll angle function defined as the sum of the

$n = 1$ term of expression (62) plus some modified Gegenbauer terms of the form of expression (60) with $n > 1$. A shape obtained from a sum of terms of this kind will be referred to for simplicity as a series shape.

The definitions of the modified Gegenbauer terms ensure that the first and second derivatives of roll angle with respect to distance will be zero at each end of a series shape. As long as $m > 1$ the same will be true of the third derivative of the roll angle with respect to distance which means, in light of the discussion at the end of Section 2 above, that the first derivative of the curvature of the track with respect to distance will be zero at each end of the shape and therefore free of discontinuity.

This Section concludes with the observation that any polynomial roll acceleration function that has zeros of the same order at both ends and zero average value so that it produces zero net change in roll velocity (and that is therefore suitable for defining a track shape) can be expressed as modified Gegenbauer series. This observation follows from the orthogonality feature of the Gegenbauer polynomials. Given a roll acceleration function accel(s) with zeros of order m at $s = -a$ and $s = a$, define coefficients $accel_n^m$ for $n >= 1$ by the formula

$$accel_n^m = \frac{1}{a^{2m+1} \cdot h_n^m} \cdot \int_{-a}^{a} dy \cdot accel(y) \cdot C_n^{(m+1/2)}\left(y/a\right) \ . \qquad (63)$$

where m is the order of the zeros of the acceleration function accel(y) at $y = -a$ and $y = a$. The normalization constant is given by the standard formula

$$h_n^m = \frac{\pi \cdot \Gamma(n+2m+1)}{2^{2 \cdot m} \cdot n! \cdot (n+m+1/2) \cdot \Gamma^2(m+1/2)} \qquad (64)$$

The original roll acceleration function, $accel(s)$, is then reproduced by the series

$$\left(a^2 - s^2\right)^m \cdot \sum_{n=1}^{N} accel_n^m \cdot C_n^{(m+1/2)}(s/a) \qquad (65)$$

where N is the largest value of n for which equation (63) gives a non-zero coefficient.

While this series representation uses Gegenbauer functions, it is not a normal Gegenbauer series expansion, and it cannot be used to represent arbitrary polynomials.

As an example, the modified Gegenbauer series representation for the roll acceleration function

$$\frac{d^2}{ds^2} r\_angle(s) \quad = \quad -k\ (a+s)^3\ (a-s)^3\ s^7 \qquad (66)$$

(taken from reference 2 and representing a form of spiral) can be expressed as the sum of four modified Gegenbauer polynomial based terms with $m = 3$ and with orders n = 1, 3, 5, and 7. The coefficients of the four terms are $\frac{-k \cdot a^7}{143}$ times 1.0, 0.50980, 0.14035, and 0.01651 respectively

10. Using spiral, Bend, and Jog combinations to upgrade curves with inadequate offset

When track engineers consider what needs to be done to existing rail lines to allow authorized speeds to be increased they often find a need to lengthen transition spirals between pairs of adjacent arcs, each pair consisting of a tangent and a curve or two reverse or progressive curves. For pure spirals of a given traditional or improved type, an increase of spiral length requires a corresponding increase of the offset of the arc pair. Sometimes the offset can be increased by a lateral shift of the alignment of an entire curve. When it is not practical to shift the curve of a tangent to curve transition it has been a common practice to replace part of the tangent by a small curve of some sort opposite in direction to the main curve and to increase the offset in that way. (See for example the article by H. Baluch, reference 7.) When this latter approach is the one selected, the shape that will be most attractive from a dynamic point of view will be a a hybrid spiral whose roll motion is the sum of two Gegenbauer based components, one an improved spiral compoent, and the other a component for a Bend away from the direction of the main curve. The present Section explains how the hybrid shape is calculated for this situation and illustrates a sample application.

The Bend example of Section 4, the Jog example of Section 7, and the Wiggle example of Section 8 have the common feature that in each one the transition shape is bordered at each end by tangent track. As a result, in those examples the non-standard relationship between track curvature and bank angle inherent in the simplified treatment is confined to the interiors of those transitions and would not necessarily affect the relationship between track curvature and bank angle of ordinary curves. We now look at the situation that the transition shape connects on at least one end to an ordinary curve. If the simplified method were applied in this situation there would be a need, in principle, to alter the relationship expressed by equation (1) between track curvature and bank angle throughout the rail system. The associated change in bank angle is small so that that would be a possibility. However, it is an unattractive possibility both because of basic physics and because of the costs of making changes to existing standards and track. Conversely, while the simplified procedure can be attractive for exploratory studies, when the time comes for engineering design the full treatment can be applied just as easily as the simplified treatment, and that is true both for the current situation and for the situations of the previous examples. The example in this Section will therefore be based mainly on the full treatment, and some figures computed with the simplified treatment will be included just for comparison.

Presentation of this example begins with an elaboration of equation (1). Denoting the overhang in the curve by *o_hang_c* and denoting the track bank angle in the curve by *r_angle_c* we can write

$$o\_hang\_c = h \cdot \sin(r\_angle\_c) \tag{67}$$

Then pursuant to equation (1) *r_angle_c* is obtained numerically as the solution of the transcendental equation

$$r\_angle\_c = \arctan\left(\frac{v^2}{g \cdot \left(radius\_c \;-\; h \cdot \sin(r\_angle\_c) \right)} \right) \tag{68}$$

(*r_angle_c* differs very little from the value that it would have with $h = 0$ so that equation (68) can be solved by iteration that converges very rapidly.).

The roll acceleration function has the form presented in Section 9 above with the parameter m set to 2, namely

$$r\_accel(s) = (a^2 - s^2)^2 \cdot \sum_1^2 j_n \cdot C_n^{5/2}(s/a) \tag{69}$$

Since the expression on the right of equation (69) is just a polynomial, the first two integrations, namely

$$r\_veloc(s) = \int_{-a}^{s} dt \;\; r\_accel(t) \tag{70}$$

and

$$r\_angle(s) = \int_{-a}^{s} dt \;\; r\_veloc(t) \tag{71}$$

can easily be done in closed form. The constant of integration for *r_angle(s)* is zero because the example is for a transition that start from a tangent.

Evaluating the preceding integral at s = a, the expression for the roll angle at the end of the shape is found to be

$$r\_angle(a) = \frac{-16\, a^6 \cdot j_1}{21} \tag{72}$$

a result that illustrates the fact that the $n = 2$ term (i.e., the Bend component) makes zero contribution to the net change in roll angle over the length of the shape.

A shape that provides a transition between a tangent and a curve is subject to two primary constraints. One is that the roll angle at the end of the shape match the roll angle, *r_angle_c,* of the curve. Solving equation (72) for j_1 this constraint gives

$$j_1 = \frac{-21 \cdot r\_angle\_c}{16\;\, a^6} \tag{73}$$

With j_1 replaced via equation (73) the remaining parameters of the shape are j_2 and a. For the purposes of this example we will set $j_2 = 0.876 \cdot j_1$ whereupon the shape will depend just on a. (There is nothing magical about 0.876; it just happened to be convenient for incedental reasons.)

The integrals to obtain the compass bearing and coordinates on the path of the roll axis are

$$b\_axis(s) = b\_axis\_init + \frac{g}{v^2}\int_{-a}^{s} dt \;\; \tan(r\_angle(s)) \tag{74}$$

,

$$y\_axis(s) = y\_axis\_init + \int_{-a}^{s} dt \;\; \sin\big(b\_axis(t)\big) \tag{75}$$

, and

$$x\_axis(s) = x\_axis\_init + \int_{-a}^{s} dt \;\; \cos\big(b\_axis(t)\big) \tag{76}$$

The second primary constraint is that the tangent to curve offset implied by the transition shape must agree with the specified offset. To obtain the offset implied by a tentative form of the shape we conceptually connect the curve to the end of the shape, see where that places the center of the curve, determine the distance from that location for the curve center to an extension of the tangent, and subtract the radius of the curve from that distance. The resultant formula for the computed offset as a function of the half length, a, is

$$offset(a) = y\_axis(a) + \cos\big(b\_axis(a)\big)\cdot(radius\_c - o\_hang\_end) - radius\_c \qquad (77)$$

An iterative search procedure is used to find the value the half length a that causes the offset computed per equation (77) to agree with the specified offset.

The illustration is for a transition between a tangent and a curve with curvature of 1.0 degree per 100 ft. chord that is banked for a balancing speed of 90.0 mph and that is offset from the tangent by 1.0 ft. The roll axis is located 8.0 feet above the track and the value used for the acceleration of gravity is 32.17 ft/sec^2.

The spiral plus Bend shape obtained for this situation by applying the full treatment with $j_2 = 0.876 \cdot j_1$ is illustrated in Figure 23. Figures 24, 25, and 26 show the similar shape obtained via the simplified method for the same parameters.

In order to allow a comparison to illustrate the beneficial effect of adding the Bend component to the improved spiral, Figures 27 through 30 are the counterparts of Figures 23 through 26 except that they have $j_2 = 0.0$ so that they show plain improved spirals.

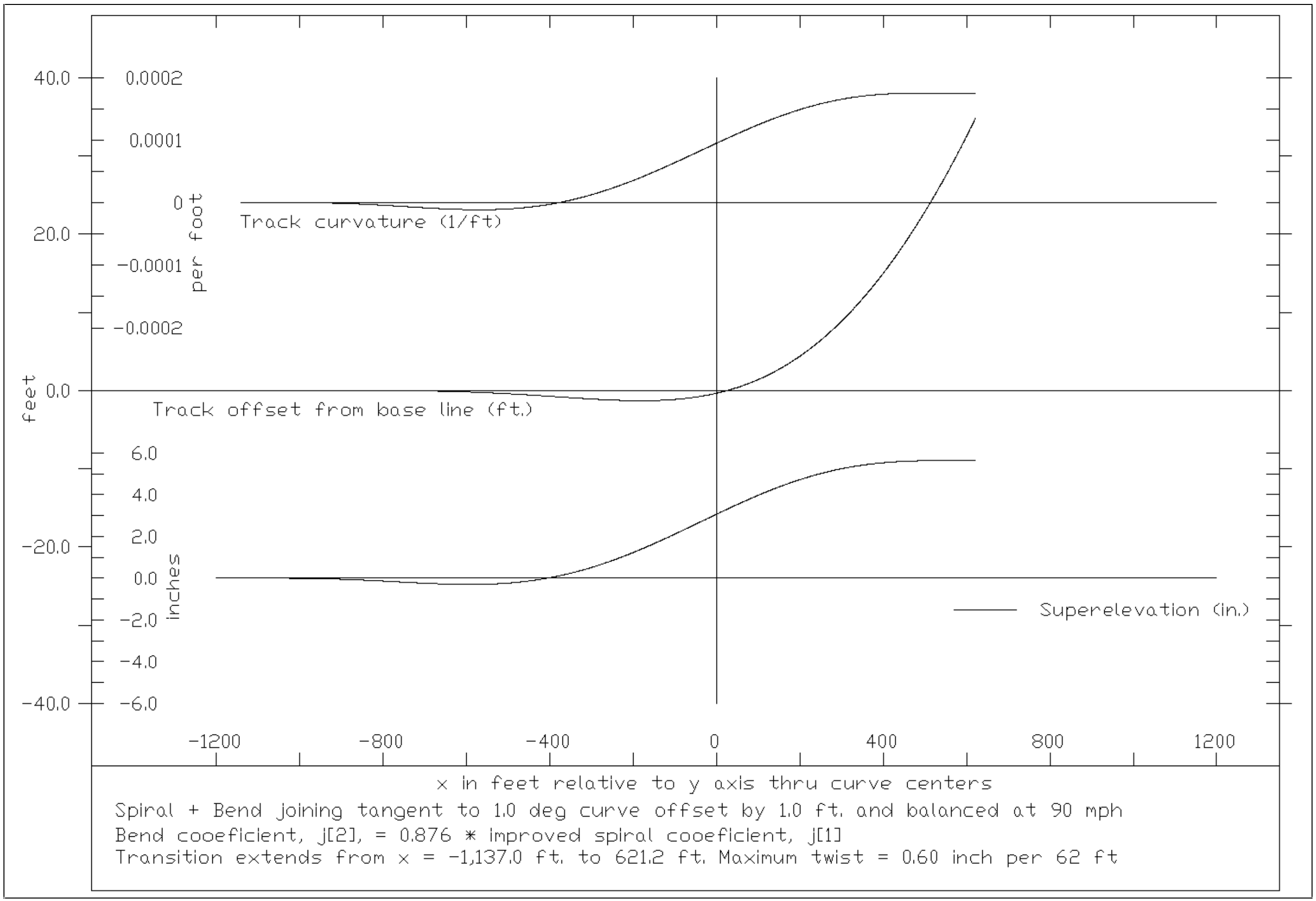

FIGURE 23. Illustration of a hybrid shape that combines improved spiral and Bend components. This shape provides a transition from a tangent to a curve and is about twice as long as a a corresponding transition consisting just of an ordinary improved spiral. A shape of this type is useful when the offset is so small that with just an improved spiral the track twist would be too large or the authorized speed could not be raised to the extent desired.

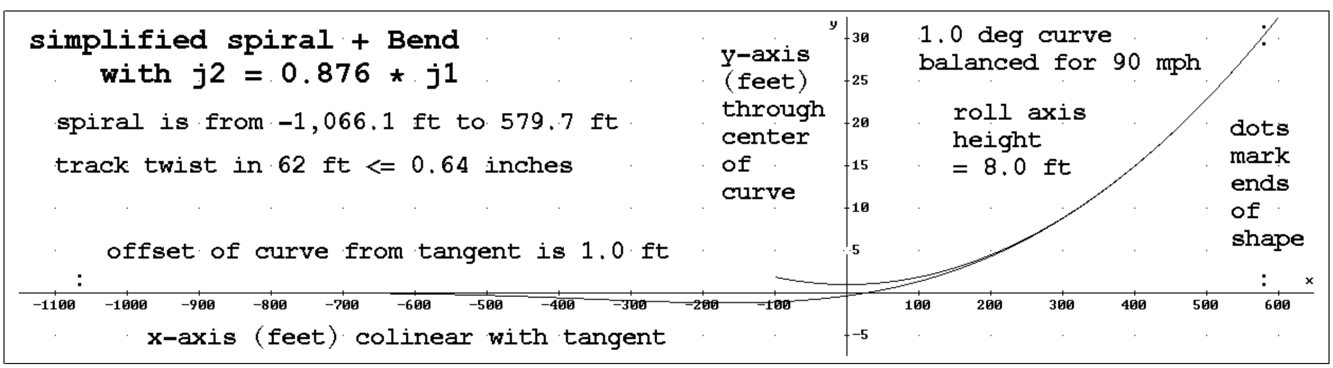

FIGURE 24. Illustration of a hybrid shape just like that of Figure 23 but constructed using the simplified treatment rather than the full treatment.

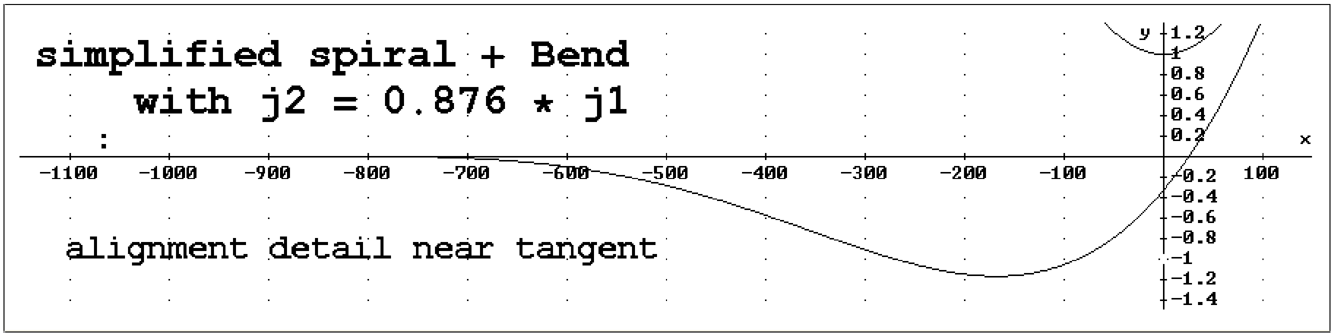

FIGURE 25. Detail of behavior of alignment of Figure 24 near the tangent.

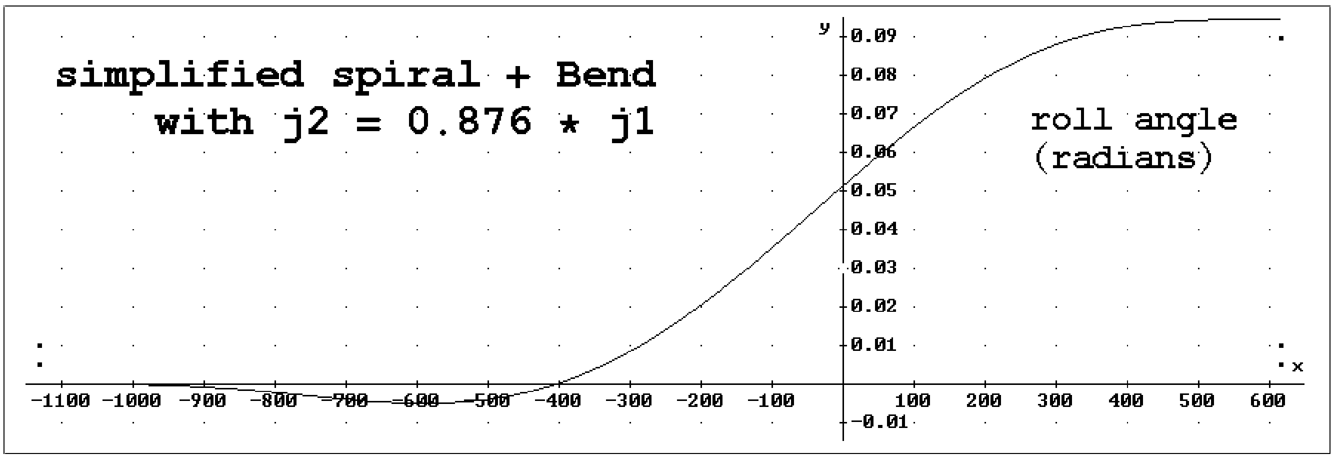

FIGURE 26. The roll angle function corresponding to Figure 24.

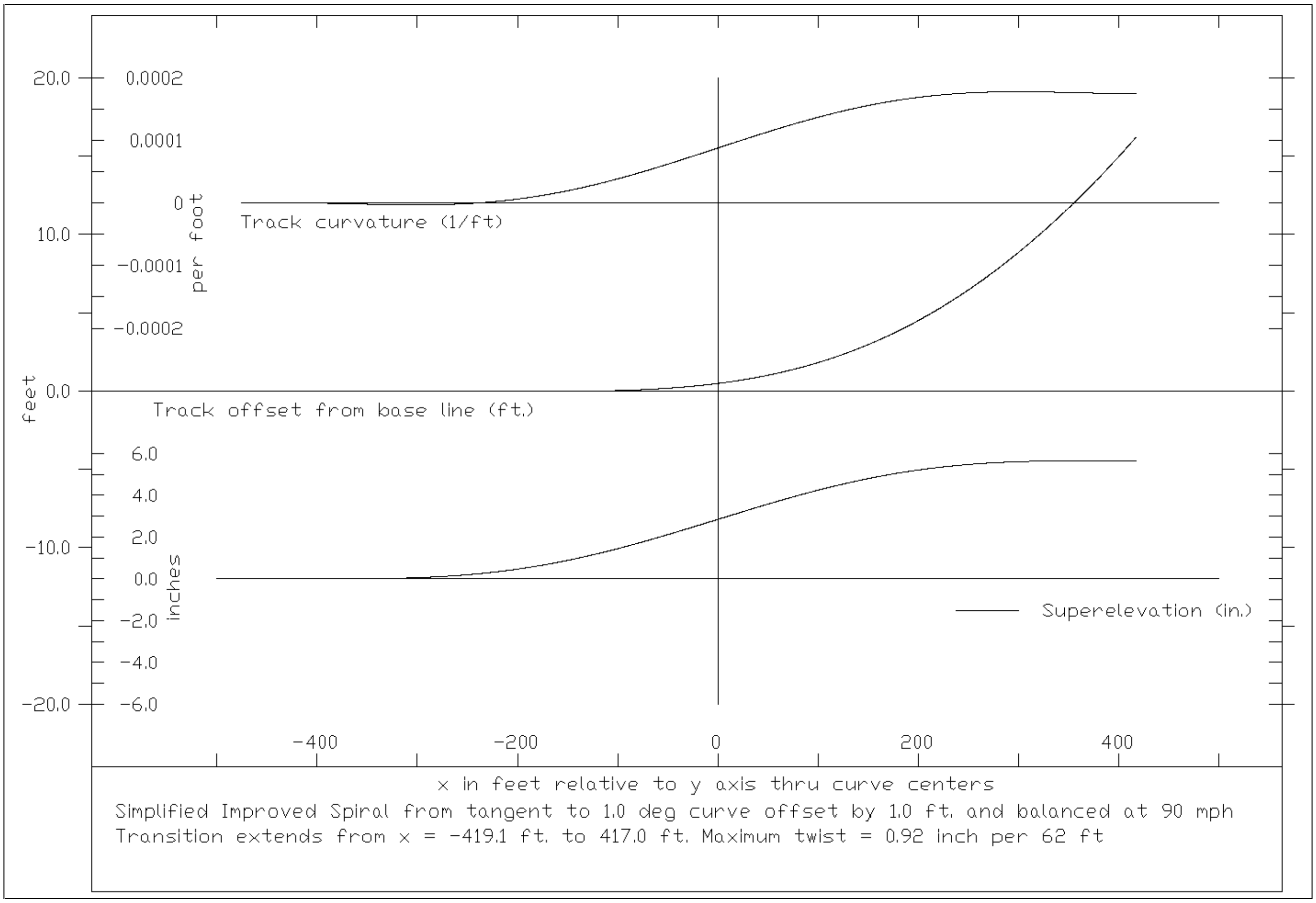

FIGURE 27. Illustration of a plain improved spiral joining the same tangent and curve as in Figure 23.

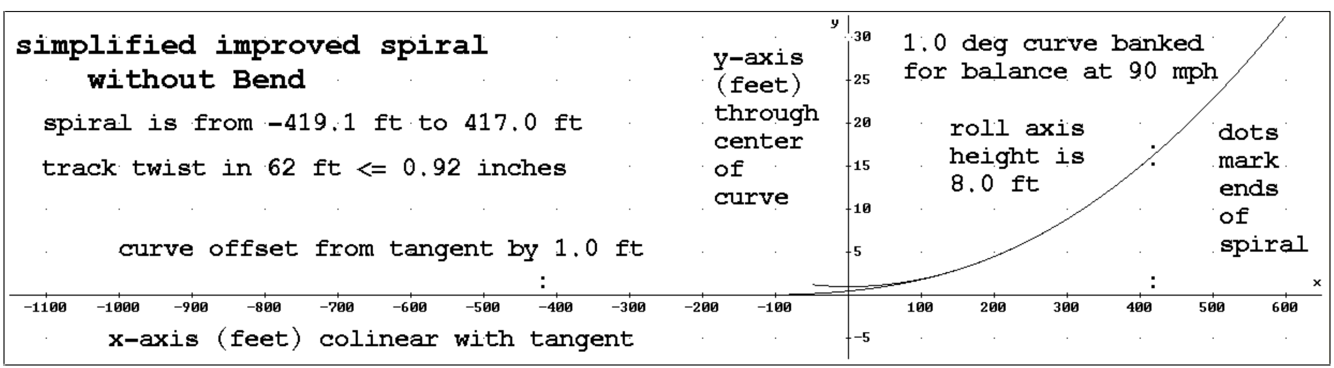

FIGURE 28. Illustration of a plain improved spiral just like that of Figure 27 but constructed using the simplified treatment rather than the full treatment.

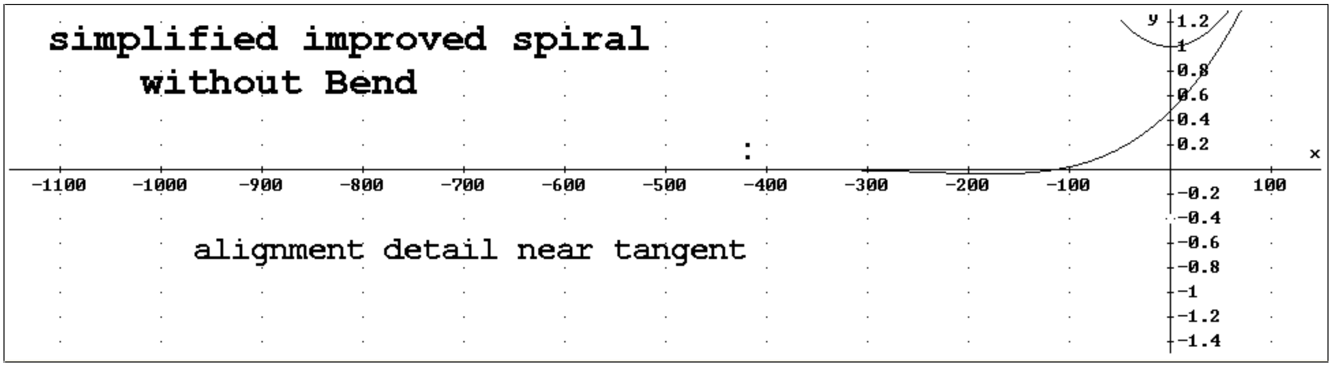

FIGURE 29. Detail of behavior of alignment of Figure 28 near the tangent.

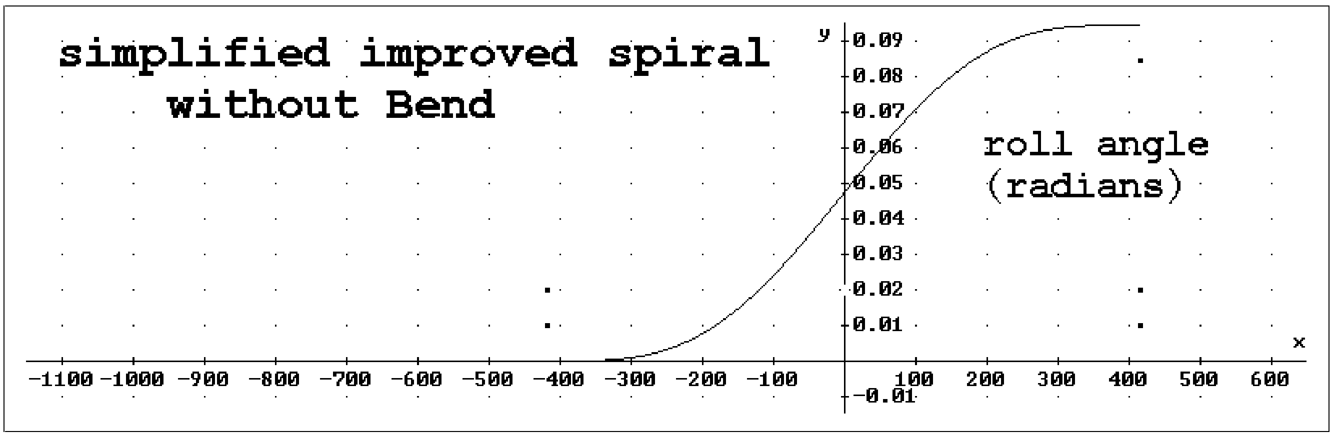

FIGURE 30. The roll angle function corresponding to Figure 28.

The objective in this Section is to get an idea of how well combinations of spirals plus Bends can alleviate inadequate offset between curves and tangents when track is being reconfigured for higher speeds. Comparing Figures 23 and 27 it can be seen that the addition of the Bend component causes the length of the transition shape to increase from about 850 feet to about 1,760 feet (Length along the track is a little more than length along the "x" axis) and causes the maximum track twist (in inches crosslevel change per 62 feet) to drop from 0.90 to 0.60. This is the kind of benefit that was being sought, and it is believed that this kind of shape will be dynamically superior to alternative shapes that achieve a similar result.

Comparing Figures 23 and 24 one can observe that for j_2 a given multiple of j_1, in comparison to the full treatment, the simplified treatment produces a shape that is a little shorter and a maximum track twist that is a little greater.

Another way that the design challenge of this Section could be met would be to insert a separate pure Jog right ahead of the spiral so that the offset at the start of the spiral would be larger and so that an ordinary improved spiral could be used. One might therefore wonder if the addition of a Jog component to an improved spiral might produce a shape as useful as that illustrated in Figure 23. A shape derived from a roll motion composed just of modified Gegenbauer terms with $n = 1$ and $n = 3$ is indeed longer than an improved spiral with the same offset. However, it has the significant defect that there is a region within the shape in which the roll angle is greater than the roll angle of the curve. That defect can be cured by adding a Bend component to the mix, but it seems that for this situation a Bend component is altogether more effective than a Jog component. It may also be noted that when a Bend and a Jog are components of a shape with a given length, the Bend will have an effective wave length that is twice that of the Jog. As a result, for a given transition shape length and vehicle speed, the vehicle suspension oscillation stimulated by a Bend will tend to be less than the oscillation stimulated by a Jog.

11. Classification of adjacent arc relationships

We have found a way to construct a variety of shapes with good dynamic characteristics. We have observed some alignment design problems for which such shapes can provide solutions. We want next to look at the ways that neighboring arcs can be related geometrically as it affects the types of shapes needed for connecting them. Such a geometrical relationship will be referred to as an Adjacent Arc Configuration. In this Section we are interested only in how connecting shapes are affected by the relationships between adjacent arcs and not in how shapes can be modified so as to avoid obstructions.

We begin by looking at the three Adjacent Arc Configurations in which spirals have traditionally been applied as illustrated in Figure 37.

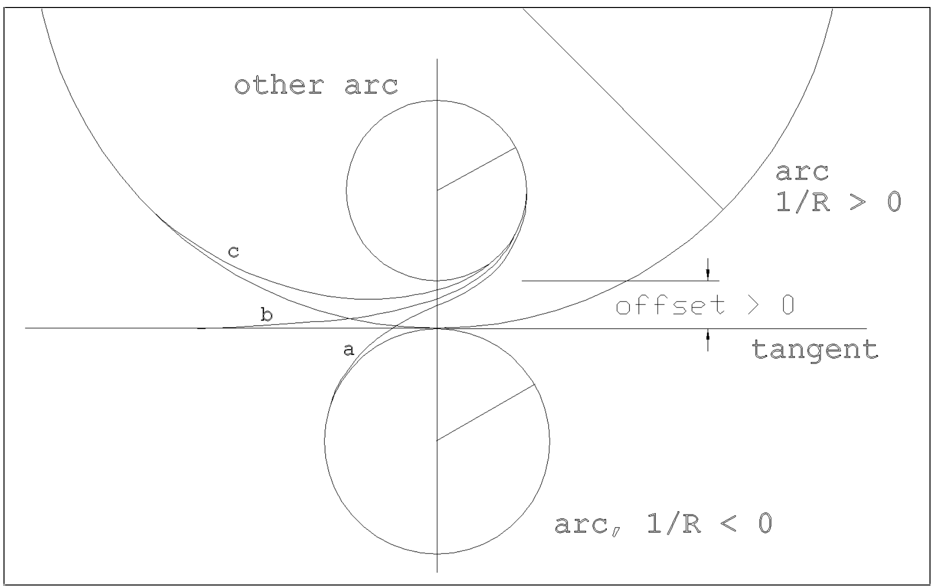

Figure 37. Illustration of the three traditional Adjacent Arc Configurations. Here one arc has positive, zero, or negative curvature, the other arc has positive curvature, and the offset between the two arcs is positive. (Shapes shown in figures in this Section are illustrative and are hand drawn rather than calculated.)

In Figure 37 the spiral labeled **a** connects reverse curves, the spiral labeled **b** connects a tangent to a curve, and the spiral labeled **c** connects progressive curves. It is characteristic of a pure spiral that the curvature (or more accurately, the curvature of the path of the track roll axis) varies monotonically with distance. In order for an Adjacent Arc Configuration to be connected by a pure spiral it must have positive offset.

Figure 38 illustrates three Adjacent Arc Configurations that also have positive offset but in which the connection to the “other” (greater curvature) arc is opposite to the normal connection.

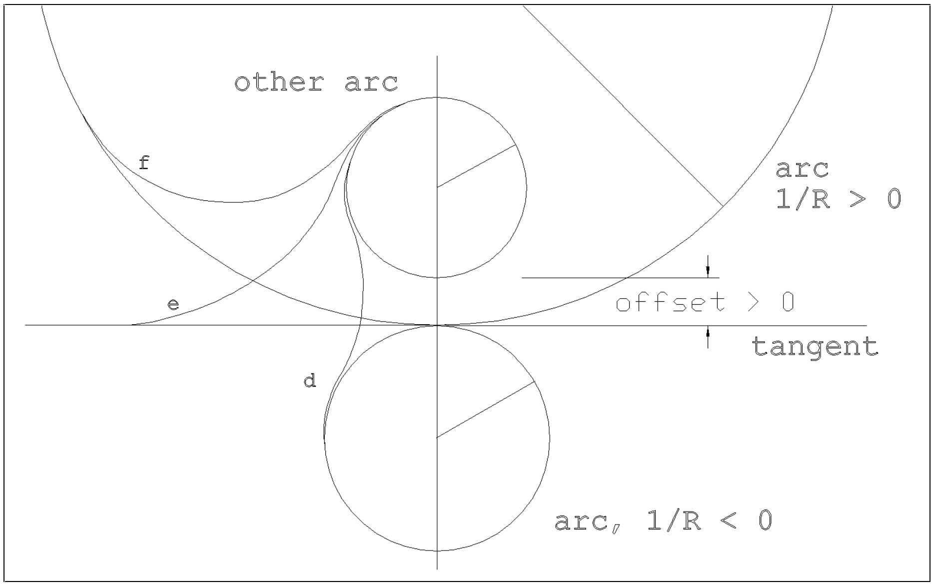

Figure 38. Illustration of three non traditional Adjacent Arc Configurations with positive offset.

In Figure 38 the curvature of each shape changes sign once more than the corresponding shape of Figure 37. That suggests that each shape can be realized by a sum of $n = 1$ and $n = 2$ terms. In visual terms and thinking of movement from left to right in the figure, each shape would be a combination of a spiral to the right and a Bend to the left.

Figure 39 illustrates additional non traditional Adjacent Arc Configurations that arise when the offset between the two arcs is negative.

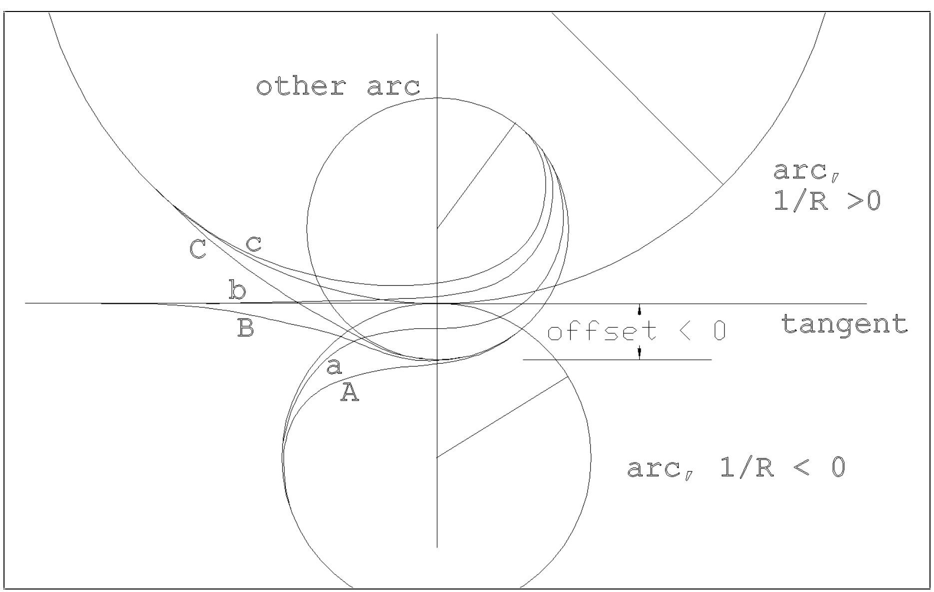

Figure 39. Adjacent Arc Configurations with negative offset.

Referring back to Figure 37 where the offset is positive it can be seen that for each of the three shapes shown there is only one natural path. The situation is different when the offset between the two arcs is negative as illustrated in Figure 39. If the connecting shape is required to stay outside of the larger curvature arc it will look respectively like **A** , **B** , or **C**. If the connecting shape is required to stay above the smaller curvature arc it will look like **b** or **c**. If neither of those constraints is imposed the connecting shape can look like **a**. If the two arcs have curvatures that are equal and opposite then shape **a** would seem the most natural. If the curvature of the smaller curvature arc is smaller by a sufficient margin then shape **A** will have curvature whose magnitude is nowhere greater than that of the "other" (larger curvature) arc. Thus it appears reasonable to consider the shapes that are applicable with negative offset to be **A**, **B**, and **C** with the proviso that **A** should be replaced by a shape of type **a** that minimizes the maximum curvature of the shape when the curvatures of the two arcs are so close in magnitude that shape **A** does not have that property.

It should be noted that the offset does not need to be negative in order for connecting shapes that look like **A** , **B** , and **C** to be useful. Shapes like **A** , **B** , and **C** will also be useful for Figure 37 type Configurations where the offset is positive but inadequate with the result that shapes of the type shown in Figure 37 would be too short to allow limits on track warp and roll acceleration to be satisfied. By the same token the offset boundary value that determines whether a particular Adjacent Arc Configuration should be associated with Figure 37 or 39 is not really zero but will rather be some positive value. What that positive boundary value is will depend on circumstances and particularly on the design speed of the track.

A figure like Figure 38 but with negative offset is not included because the geometrical properties of the connecting shapes would be the same as those already shown in Figure 38.

Figure 40 illustrates the connecting shape that applies when the adjacent arcs are non-parallel tangents.

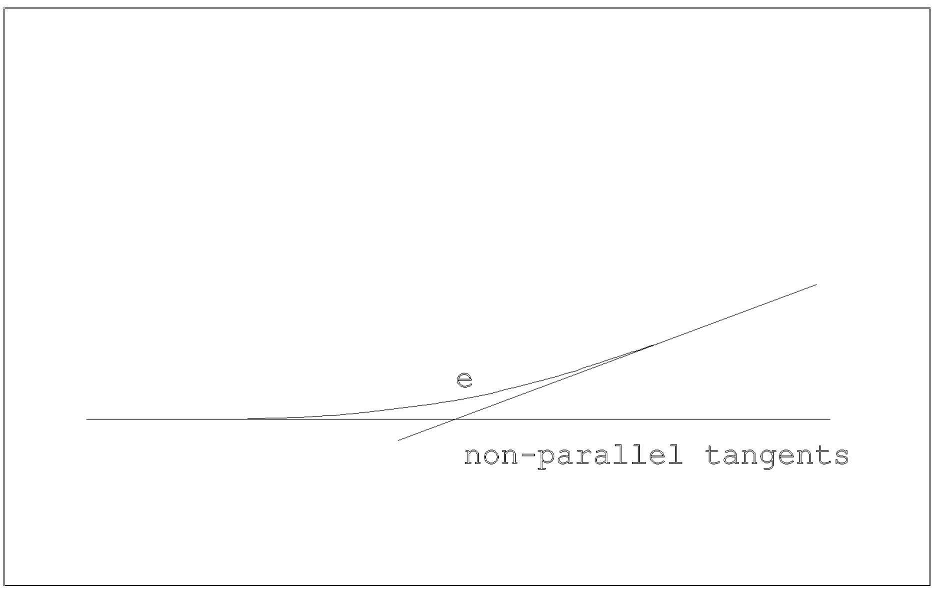

Figure 40. Illustration of shape connecting non-parallel tangents.

The shape labeled **e** in Figure 40 is a Bend. It corresponds to the shape labeled **e** in Figure 38 if as the radius of the "other arc" goes to infinity its center moves not upward but rather to the right and downward, and the offset goes to minus infinity. This is the only Adjacent Arc Configuration in which the offset is undefined.

Figure 41 illustrates the connecting shapes that are applicable when the two arcs are both tangents.

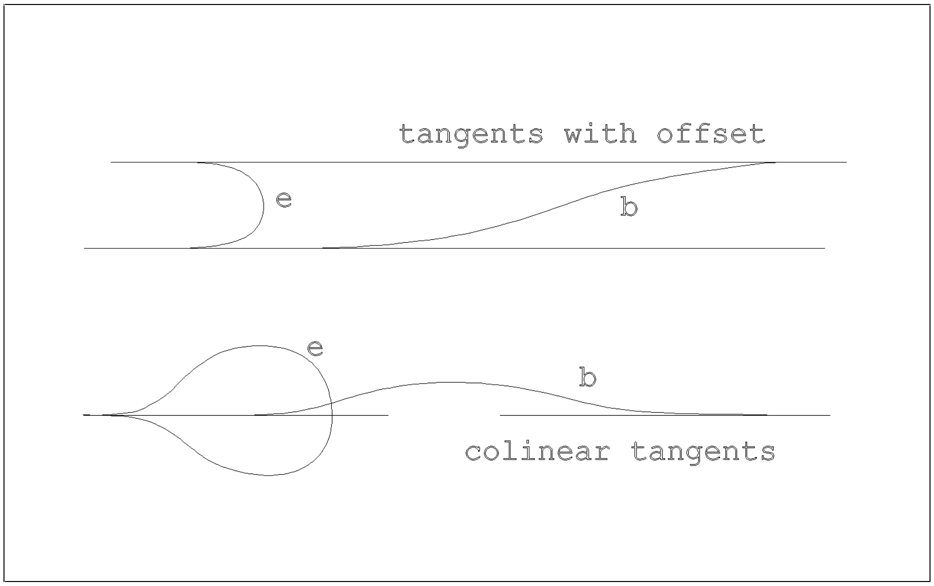

Figure 41. Illustrations of shapes to connect neighboring tangents.

The upper part of Figure 41 is for tangents with offset. The lower tangent corresponds to the tangent of Figures 37 and 38 and the upper tangent corresponds to the upper arc of Figures 37 and 38 but with its curvature reduced to zero. Shapes **b** and **e** correspond to the identically labeled shapes of Figures 37 and 38. Shape **b** is a Jog and is the recommended form for crossovers and turnouts. Shape **e** is a Bend.

The lower part of Figure 41 is for collinear tangents. The preceding comments apply here as well with the exception that in order to be non-trivial shape **b** needs to change from a Jog to a Wiggle and shape **e** is a Bend with some Wiggle added.

Whereas it would be reasonable to use shape **e** of Figure 38 in practice, it does not seem very likely that either of the shapes labeled **e** of Figure 41 would be used in practice. (The practical solution for such large angle turns would likely be a spiral – curve – spiral sequence.) Still, it is desirable to take note of all of the possibilities.

12. Modified Gegenbauer series for alignments that avoid obstacles

This Section outlines a procedure for calculating a series shape that circumvents fixed obstructions and passes to the left or the right of each one as specified. The ordinary boundary constraints (unrelated to obstructions) depend on the adjacent arcs and are as indicated Section 4 for Bends, in Section 7 for Jogs, in Section 8 for Wiggles, and in Section 10 for spirals.

The steps are as follows:

[1] – Use the first primary constraint to obtain the lowest order modified Gegenbauer series coefficient as a function of the shape half-length parameter, a. Choose initial estimates for remaining series coefficients as multiples of the lowest order coefficient. Include terms of order up to a user specified limiting order denoted N. Also choose an initial estimate for the half-length, a.

[2] – Evaluate the integrals that give the bearing and coordinates along the path of the roll axis.

[3] –Determine the coordinates of points along the track taking account of the bearing angle, the roll angle, and the roll axis height.

[4] – Using the track shape that results from the current estimates for the half-length and the series weights, determine the amount by which the computed alignment fails to satisfy the remaining constraints and the distance by which the computed alignment transgresses on the wrong side of each obstruction point.

[5] –Repeat steps 1 through 4 above with the parameter values varied one at a time so that the dependence of the errors on each parameter can be estimated.

[9] – Compute improved estimates of the parameter values and then repeat steps 1 through 5 above. Continue until a series shape that satisfies the constraints has been found or it is determined that a solution is not possible with the current value of N.

[10] If a solution could not found with one value of N, increase N and try again. Conversely, if a solution was found for the first choice of N, then reduce N and see if a solution with a lower value of N can be found.

13. Modified Gegenbauer series for maintenance of track geometry

In track with traditional spirals the defective dynamics of the traditional spiral geometry causes the vehicles to apply systematic net lateral forces to the track structure near the ends of the spirals (see references 3 and 4 for examples). This causes slow but progressive lateral movement of the track. When track has become rough due to traffic (and other factors) its geometry is corrected as a part of periodic maintenance. In recent years this maintenance has been done mainly using large tamping machines that make corrections determined by computer programsThere are two different ways that alignment can be improved. One way is to bring the alignment back into conformance with engineering drawings that specify absolute locations through which the rails should pass. The other way is to make the alignment smooth relative to a running average of the current location of the track. Where the track is not anchored by fixed objects such as bridges it has been the general practice in North America to do smoothing relative to the existing alignment of the track and not to try to bring the track back into conformance with engineering drawings.

The net result is that there are many heavily used main line railroad curves whose alignments at the ends of spirals deviate by distances such as 5 and 10 inches from where they are supposed to be according to drawings. Maintenance by smoothing keeps such curves usable. It would probably not be considered cost effective to bring such curve alignments back into conformance with traditional drawings except when they lie on lines that are being completely reconstructed. It would therefore be useful to have a method for determining a set of lateral shifts that produce a corrected alignment with optimal dynamic properties subject to the limitation that no single shift exceen a given limit, such as 3.0 inches. If the corrected alignment produced by the shifts were specified mathematically relative to absolute coordinates it could serve to define an accepted shape for the track, and subsequent alignment maintenance could be programmed to preserve (or further improve) that shape and prevent further drift. Even if the documentation, engineering, and control infrastruture necessary to maintain absolute track alignments were not available, the method would still offer the benefit of providing optimal corrected alignments subject to a limit on allowed track shift. It is important to keep in mind that by including improved spirals, the corrected alignments would not be subject to the systematic lateral forces caused by the defective dynamics of traditional linear spirals and also caused to a lesser extent by other traditional spirals. The tendency toward drift will therefore be much reduced even without absolute location control during tamping.

The series shapes based on modified Gegenbauer polynomial terms as set forth herein appear to provide the basis for such a method. A procedure for finding a modified Gegenbauer series to define an accepted geometry for a track segment such as a spiral and some track past each end thereof in which departure from ideal geometry is substantial is as follows:

[1] – Obtain measured chord offset values at stations spaced uniformly along the track segment being analyzed. From the offsets construct the x & y coordinates of each station point with respect to a Cartesian coordinate system of convenience. (If accurate track line survey data are available, then those data can be used as a check or, if they are very precise and accurate, as a substitute.)

[2] – Once the existing station point coordinates are known, a series shape that provides an approximation for a segment of the existing track line in which curvature is to vary with distance can be obtained via the procedure of Section 12 above with a modification as follows. Enter the existing station points as though they were obstructions but modify the error criteria so that the shape is allowed to pass on either side of each existing point and take the RMS value of the distances of the existing points from the series shape as a measure of the cumulative error. The values of the $j[\mathrm{n}]$ coefficients that minimize the cumulative error for a given value of N can be determined by iterative search as outlined in the preceding Section. Once an optimal set of $j[\mathrm{n}]$ coefficient values has been obtained the distances from eacj existing station point to the series shape can be determined. If any of those distances exceeds that maximum allowed throw value, then a judgment can be made as to whether to mark that point as an obstruction and treat it according to the procedure of Section 12 or to employ a larger track shift at its location. The final series shape is then used to calculate the track shifts that a tamping machine will be instructed to make.

A compromise needs to be found between a lower value of N which will give better dynamics but require larger track throws and a higher value of N that will allow more shape oscillation in the final alignment but whose corresponding track throws will not be as large. An estimate of an upper limit for N can be made as follows. The number of cycles in a Gegenbauer polynomial of order n is $(n + 1)/2$. Taking account of the length of the transition $2a$, the maximum authorized speed v, the highest frequency of excitation produced by the series shape will be $v\,(N - 1)/(4\,a)$. As a rule of thumb this frequency should be less than about one third the lowest vehicle secondary suspension resonant frequency. Until experience has been gained vehicle operation over candidate shapes should be simulated.

There is an alternate way to obtain a series shape to approximate and document an imperfectly corrected transition alignment. The alternate method is algorithmically more complex but may still be useful. The remainder of this Section outlines the alternate method.

[1] – Same as [1] above.

[2] – From the station point coordinates construct a table of values of curvature as a function of distance along the transition. Taking account of the maximum speed that will be authorized, use equation (1) to obtain corresponding track roll angle values at the station points along the transition.

[3] – Obtain a modified Gegenbauer series approximation for the track roll as a function of distance using the procedure described at the end of Section 9 above but adding another term analogous to expression (62) to represent the spiral component if the shape needs to include a spiral contribution.

The curvature of the series approximation is obtained from the roll function via equation (1).

[4] – Integrate equations (74) through (76) to obtain the compass bearing and coordinates of the path of the roll axis as a function of distance. Determine the coordinates of point on the track taking account of the bearing, the roll angle, and the roll axis height.

[5] – Compare the track shape thus obtained with the original shape of step [1] above. Correct for possible systematic errors in the measured offsets, and for processing errors. Do so by adjusting the series coefficients for $n = 1$ and 2 so that the total change in roll angle and compass bearing have the correct values.

76 – Calculate the corrective track shifts that a tamper would be directed to make based on the differences between the existing alignment from step [1] and the current series shape. If the computed track throws are larger than desired, then increase N and repeat. If the computed track throws are smaller than throws that have been found practical in the past, then reduce N and repeat. Based on manual review, look for corrections that should be allowed even though they exceed the desired track shift limit.

14. Conclusions

The Bend, Jog, and Wiggle shapes defined herein will allow the design of alignments with dynamic characteristics that are better than those of corresponding alignments incorporating only tangents, spirals, and arcs. Quantifying the character and magnitude of such improvements will be a task for future simulations analogous to those reported in reference 3 and for field tests.

It is convenient to use Gegenbauer polynomials to represent combinations of spirals, Bends, Jogs, and Wiggles. Such combinations can be used in new designs to find geometry that avoids obstructions and can be used to document and maintain non-ideal curve geometry of existing tracks whose shapes have become deformed over the course of time. The roll acceleration functions presented herein for defining Bends, Jogs, and Wiggles are not the only possibilities. However, the functions defined herein are expected to be preferred over other possibilities in most cases.

It is expected that Jogs will be widely applied for improvement of the geometry of turnouts and crossovers and that spiral plus Jog combinations will be widely applied in upgrading of spirals with inadequate offsets to allow for higher speeds. It is also expected that modified Gegenbauer series shapes as defined herein will be widely used to improve the track alignment results achieved by tamping machines.

The shapes described herein can bring about improvements not only when applied to railroad tracks but also when used in design of geometry for maglev guide ways, highways, roller coasters, and bob sled runs.

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