AN IMPROVED ANALYSIS FOR THE DETERMINATION OF REQUIRED BALLAST DEPTH

by

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SUMMARY

In railway engineering, in order to reduce the rate of subgrade deterioration, there is a need to know the vertical normal stress distribution, $\sigma_{zz}(z)$, through the ballast for the determination of adequate ballast depth between the tie bottoms and the subgrade surface. According to the AREA Manual (Section 22, Part 3), the required ballast depth, $z_{req}$, should be such that $\sigma_{zz}(z_{req}) \leq 20 \text{ lb/} \text{in}^2$.

Because subgrade and ballast conditions vary along a track, for railroad engineering purposes it is sufficient to use a simple analysis that will predict the vertical stresses below the tie-bottoms with a reasonable accuracy. Although during the past several decades, a number of analyses were proposed, to date, there is no generally accepted method for determining these stresses. The purpose of this paper is to present a simple method for problems of this type.

1. INTRODUCTION AND STATEMENT OF PROBLEM

The proposed method is based on the well-known solution by Boussinesq (1885) for a vertical point load that acts on the free surface of a homogeneous, isotropic, weightless, elastic half-space. Subsequently, the vertical stresses caused by other loads that act on the free surface, were determined from it, using superposition. Solutions for various load types are presented in texts on soil mechanics and foundation engineering, such as Kollbrunner (1946, Vol. I), Florin (1959, Vol. I), Terzaghi and Peck (1967, Chapter 6), Bowles (1988, Chapter 5), and Das (1990, Chapter 3).

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These methods of analysis are cumbersome and time-consuming for the estimation of vertical stresses in the ballast layer caused by several loaded ties. Therefore, design charts will be produced for a range of track parameters such as the tie width $b$, effective tie length $l$, center-to-center tie spacing $a$, and the vertical track stiffness.

For the utilization of these design charts, only the maximum pressures under the ties have to be known. They will be determined by modeling the rail-tie-ballast structure as a beam on a Winkler foundation, which is widely accepted in railway engineering.

Another method for the determination of vertical stresses in ballast was derived empirically from test data. It is the Talbot formula (1920), listed in the AREA Manual for Railway Engineering (1991, Chapter 22, Section 3.3). A review of the original Talbot Committee report (1920) suggests that its use as listed in the AREA Manual differs from the one originally described by Talbot. This and related issues are discussed, and the results of the Talbot formula are compared with the produced design charts.

The generated design charts are then compared with the experimental data by Salem (1966). It was found that the agreement is close, indicating that the charts are suitable for design purposes. The paper concludes with a practical application of the recommended charts, for the determination of an adequate ballast depth.

2. ANALYTICAL APPROACH FOR THE DETERMINATION OF $\sigma_{zz}$

A comprehensive analysis of the rail-tie-ballast structure should determine simultaneously the response and stresses of all track components (i.e. rails, tie-plates, ties, ballast, and subgrade). However, this is an extremely involved task, with many unknown variables (like the fact that the ballast and subgrade are granular materials, often cohesionless, whose densities vary along the track). Therefore, this approach is not suitable for railway engineering purposes. For this reason,
it appears sufficient to simplify the analysis of these vertical stresses, by decoupling it into two parts as follows:

1) The *first part* investigates the behavior of a rail, especially the forces it exerts on the ties and ballast, by modeling it as continuously attached to a Winkler base which consists of a layer of closely spaced springs (as recommended in AREA Manual, 1991, Chapter 2).

2) The *second part* investigates the response of the rail supporting base, which consists of the tie-plates, ties, ballast, and subgrade by utilizing the rail pressures obtained from the *first part*.

The analyses to be presented assume the validity of the classical beam bending theory for the rail, that the effect of axial forces in the rail is negligible ($N=0$) on the determination of the vertical stresses in the ballast, and that there are no distributed moments (or torque) along the rail axis caused by axial resistances between the rail and the ties. For a justification of this analytical approach, refer to Kerr (1995).

### 3. DETERMINATION OF PRESSURE DISTRIBUTION ALONG TIE BOTTOMS USING THE DECOUPLED ANALYTICAL APPROACH

#### 3.1 Modeling of a Cross-Tie Track

As stated above, a configuration of wheel loads that yields the largest pressures at the tie bottoms has to be determined first. The physical problem is shown in Fig. 1(a). The corresponding analytical model is shown in Fig. 1(b).

The differential equation for bending of an elastic beam, used for rail analyses, is

$$EI \frac{d^4 w(x)}{dx^4} + p(x) = q(x),$$

(1)

where $x$ is the rail reference axis, $w(x)$ is the vertical deflection of the rail axis at $x$, $p(x)$ is the "continuous" distributed pressure exerted by the ties on the rail base, $q(x)$ represents the distribution of vertical wheel loads on the rail, and $EI$ is the flexural stiffness of one rail in the vertical plane. Following Winkler (1876),
\[ p(x) = k w(x), \quad (2) \]

where \( k \), the proportionality constant, is the base parameter (force/length\(^2\)) for one rail. In railway engineering, \( k \) is also referred to as the “track modulus”. With eq. (2), eq. (1) becomes

\[ EI \frac{d^4w}{dx^4} + kw = q. \quad (3) \]

This equation is suggested in the Manual of the American Railway Engineering Association (1991), Chapter 22, Part 3, for use in track analyses.

Schwedler (1882) presented a solution to eq. (3) for an infinitely long beam subjected to a concentrated vertical force, \( P \), at \( x = 0 \). It may be stated as

\[ w(x) = \frac{P \beta}{2k} e^{-\beta|x|} (\cos \beta|x| + \sin \beta|x|), \quad -\infty < x < \infty, \quad (4) \]
where \( |x| = |x| \) and

\[
\beta = \sqrt[4]{\frac{k}{4EI}}. 
\]  

(5)

Using eqs. (2) and (4), the “continuous” pressure distribution along the bottom of the rail becomes

\[
p(x) = k w(x) = \frac{PB}{2} e^{-\beta|x|} \left( \cos \beta |x| + \sin \beta |x| \right).
\]  

(6)

Note that the \( p(x) \) term represents the collective response of the ties, tie-plates, tie-pads (when used), ballast, and subgrade. Fig. 2 shows the typical normalized pressure distribution, for different \( k \)-values, along the bottom of an infinitely long 136RE rail caused by a single wheel load, \( P \).
3.2 Actual and Assumed Pressure Distributions at the Tie-Ballast Interface

In the previous section, the “continuous” pressure that the rail exerts on the ties (and vice versa) was determined. In the following, the contact pressures that the ties exert on the ballast are discussed.

After tamping, a typical pressure distribution along the bottom of the entire length of a tie, $L$, is shown in Fig. 3(a). Note that the pressure in the middle of the tie is much smaller than the pressure under each rail-seat, because the ballast is tamped mainly under the rail-seat area in order to avoid the occurrence of “center-bound ties”. This pressure distribution varies with accumulated tonnage of the passing trains. Therefore, in order to simplify the analyses, it is often assumed that the pressure exerted by the tie on the ballast is uniform and acts only in the outer thirds of the tie, as shown in Fig. 3(a). It is also assumed that the variation of pressure along the width of each tie bottom is uniform, as shown in Fig. 3(b).

Using this pressure distribution, a general configuration of pressures that acts on the ballast base can be constructed, as shown in Fig. 4. This configuration will be utilized throughout the paper when calculating the vertical stresses in the ballast.
3.3 Determination of the Uniform Pressures Along the Tie-Bottoms

Using eq. (6), typical pressure distributions \( p(x) \) along the bottom of a rail, shown in Fig. 2, are obtained. In order to estimate the pressures exerted by each tie on the ballast, the *rail-seat force*, \( F_n \), that acts on top of each tie, is determined first. Following Kerr (1995), each *rail-seat force* caused by a wheel load \( P \) is estimated by multiplying the pressure ordinate at the center of each tie width by the center-to-center tie spacing, \( a \), as shown in Fig. 5(a). That is,

\[
F_0 = p(0) \cdot a \quad ; \quad F_1 = p(a) \cdot a \quad ; \quad F_2 = p(2a) \cdot a , \quad \text{etc.,} \quad (7)
\]

where \( p(0) \), \( p(a) \), and \( p(2a) \) are the pressure ordinates at the \( x = 0 \), \( x = a \), and \( x = 2a \). Once the rail-seat forces \( F_n \) are calculated, the average pressure along each tie bottom, \( p_n \), shown in Fig. 5(b), is determined using vertical equilibrium. It is assumed that the weights of the ties are negligible compared to the tie bottom pressures caused by the wheel load, \( P \). Therefore, vertical equilibrium yields

\[
p_n = \frac{F_n}{b \cdot l} = \frac{a \cdot p(n a)}{b \cdot l} \quad n = 0,1,2K , \quad (8)
\]

where \( l = L/3 \).
4. VERTICAL STRESSES IN THE BALLAST UTILIZING THE BOUSSINESQ SOLUTION

Boussinesq (1885) determined the stresses in a homogeneous, linearly elastic half-space caused by a concentrated point load $P$, as shown in Fig. 6.

Some of the assumptions associated with this method are not satisfied for railway engineering problems. One such shortcoming is that the ballast-in-track is typically comprised of crushed stones; thus, it is neither a homogeneous nor elastic medium, and may compact differently along the length of the track. Also, the condition that there should not be slippage inside the elastic solid is generally violated in the ballast layer. Another shortcoming is that an infinitely wide medium is assumed to support the applied load. In reality, there is a finite width to the ballast section. It appears reasonable to expect that typical ballast sections with at least a 6-inch shoulder width from each end of the ties and a minimum slope of 2:1 will sufficiently
enclose the stress envelope (or pressure bulb) and not significantly change the vertical stress values computed for an infinitely wide ballast section. Also, since a typical dimension of the ballast stones may be about 1½ inches, there could possibly be only about 8 stones along the height of a ballast layer of 12 inches. Therefore, this ballast layer hardly represents a “continuous” elastic medium.

![Diagram of normal stresses at a point in a medium due to a point load](image)

**Fig. 6 Normal Stresses at a Point in a Medium Due to a Point Load**  
(shear stresses were omitted for clarity)

However, because the probabilistic approach [Kandaurov (1966), and Harr (1977)] yields similar results for the vertical stresses \( \sigma_{zz} \), the only stresses of interest for the design problem under consideration, and in the absence of a better analytical method, the Boussinesq solution will be used.

The expression derived by Boussinesq (1885) for the vertical normal stresses shown in Fig. 6 is

\[
\sigma_{zz} = \frac{3P}{2\pi} \frac{z^3}{R^5} = \frac{3P}{2\pi} \frac{z^3}{(r^2 + z^2)^{5/2}},
\]  

(9)

where
It is important to note that $\sigma_{zz}$ does not depend on either of the two elastic constants ($\nu, E$). Also, note that compression stresses are assumed positive in eq. (9).

Since the Boussinesq formulation is linear, superposition is valid. Therefore, the solution for a concentrated force $P$ can be used to obtain the $\sigma_{zz}$-values for other load distributions such as strip loads, line loads, linearly increasing loads and uniformly loaded areas. The solutions for loads of this type are generally given in texts on soil mechanics [e.g. Singh (1967), Chapter 9; Bowles (1977), Chapter 5; Spangler (1982), Chapter 16; Das (1994), Chapter 7].

To calculate the vertical normal stresses in the ballast, at any depth below a tie, the most useful of these solutions is the one which determines the stresses under the corner of a uniformly loaded rectangular area as shown in Fig. 7, which is representative of the assumed pressure (load) distribution on each outer third of the tie bottom (Fig. 3). For this particular loading, the $\sigma_{zz}$-expression in eq. (9) is a function of the uniform load intensity $p_n$, the loaded area (width $b$ and length $l$), and the depth $z$ at which the vertical stress is to be determined. It is given by Das (1994, p. 234) as

$$\sigma_{zz} = p_n I_z,$$

where

$$I_z = \frac{1}{4\pi} \left[ \frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2+m^2n^2+1} \left( \frac{m^2+n^2+2}{m^2+n^2+1} \right) + \tan^{-1} \left( \frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2-m^2n^2+1} \right) \right],$$

$$m = \frac{b}{z} \quad ; \quad n = \frac{l}{z}. \quad (13)$$

To illustrate the use of this Boussinesq solution, consider a homogeneous, semi-infinite, linearly-elastic solid that is subjected to a uniformly distributed load, $p_n$, over a rectangular area at the free surface, as shown in Fig. 7. To determine the vertical stress under the center of the area (point A), the loaded area is divided into four equal rectangles (denoted by ABCD, ADEF, AFGH, and AHJB), each having the common corner at A. Then, eqs. (11) and (12) are used to determine the vertical stress at a given depth $z$ under point A by multiplying the vertical stress
caused by one of the smaller rectangular loaded area by 4, in order to determine the stress due to the entire loaded area (rectangle CEGJ).

![Fig. 7 Breakdown of Tie Pressure Distribution For Use of Superposition](image)

When the point at which the stress is to be determined is outside of the footprint of the loaded area, another superposition scheme is utilized. For example, to determine the vertical stress at point A' that is located outside the loaded area CEGJ, consider the two rectangular areas, A'BCK and A'FEK, shown in Fig. 7. The vertical stress at A' at some specified depth $z$ caused by the loaded area CEGJ is then obtained by subtracting the influence factor ($I_z$) for rectangular area A'FEK from the $I_z$ value for the rectangular area A'BCK. Due to symmetry, the result is then multiplied by two to account for the entire loaded area CEGJ.

These superposition procedures, in conjunction with eqs. (11) and (12), are used to calculate the vertical stresses under a tie caused by a number of railroad ties, as shown in Fig. 4. The total vertical stress distribution under the center tie is determined by summing the vertical stress contributions of each tie.

As shown by Bathurst (1997), and referring to Fig. 4, the vertical stress contributions from the ties beyond each second adjacent tie, as well as from the opposite end of all ties, are
negligible. This is obvious from Fig. 8, where the vertical stress contributions from the adjacent ties decrease rapidly and are negligibly small beyond the second adjacent ties. Therefore, to determine the total vertical stress under the center of the tie bearing area, only the assumed uniform pressure over that area, and the first two adjacent effective tie areas on either side should be included.

![Total Vertical Stress Distribution and the Stress Contributions of Adjacent Ties](image)

**Fig. 8**   Total Vertical Stress Distribution and the Stress Contributions of Adjacent Ties

This finding facilitates the use of a general equation that would estimate the total vertical stresses under a tie caused by a series of loaded ties due to wheel load(s), $P$. The general equation depends upon the wheel load $P$, tie dimensions $b$ and $l$, center-to-center tie spacing $a$, and the depth $z$ at which the vertical stress is to be determined. For the configuration of tie
pressure distributions shown in Fig. 9, with the exclusion of vertical stress contributions from beyond the second adjacent tie and the opposite end of all ties as shown in Fig. 4, eqs. (11) and (12) given by Das (1994, p.234) can be rewritten, using symmetry, as follows:

\[ \sigma_{zz} = 4 p_0 I_1 + 4 p_1 \left(I_2 - I_3\right) + 4 p_2 \left(I_4 - I_5\right), \]  

where

\[ I_i = \frac{1}{4\pi} \left[ \frac{2 m_i n_i \sqrt{m_i^2 + n_i^2 + 1}}{m_i^2 + n_i^2 + m_i^2 n_i^2 + 1} \left(\frac{m_i^2 + n_i^2 + 2}{m_i^2 + n_i^2 + 1}\right) + \arctan \left(\frac{2 m_i n_i \sqrt{m_i^2 + n_i^2 + 1}}{m_i^2 + n_i^2 - m_i^2 n_i^2 + 1}\right) \right] \]

for \( i = 1, 2, 3, 4, 5 \) and for the layout of the tie pressures shown in Fig. 9,

\[ m_1 = \frac{b}{2z}, \quad m_2 = \frac{a - b/2}{z}, \quad m_3 = \frac{a + b/2}{z}, \quad m_4 = \frac{2a + b/2}{z}, \quad m_5 = \frac{2a - b/2}{z}, \]

\[ n_1 = n_2 = n_3 = n_4 = n_5 = \frac{l}{2z}. \]

Fig. 9  Modified Configuration of Tie Pressure Distributions Used in the Determination of Vertical Stresses (Plan View)
With specified track parameters, this equation can be generalized to determine the total vertical stress $\sigma_{zz}$ at any depth, that does not have a symmetric tie-ballast pressure configuration about the center tie, by expanding Eq.(14a) as

$$\sigma_{zz} = 4 \ p_0 \ I_1 + 2 \ p_{1R} (I_2 - I_3) + 2 \ p_{1L} (I_2 - I_3) + 2 \ p_{2R} (I_4 - I_5) + 2 \ p_{2L} (I_4 - I_5),$$

(15a)

where $p_0$ is the tie-ballast pressure at the bottom of the center tie (corresponding to the point of maximum pressure under the rail), $p_{1R}$ is the tie-ballast pressure at the bottom of the first adjacent tie to the right, $p_{1L}$ is the tie-ballast pressure at the bottom of the first adjacent tie to the left, $p_{2R}$ is the tie-ballast pressure at the bottom of the second adjacent tie to the right, $p_{2L}$ is the tie-ballast pressure at the bottom of the second adjacent tie to the left, and

$$I_i = \frac{1}{4\pi} \left[ \frac{2m_i n_i \sqrt{m_i^2 + n_i^2 + 1}}{m_i^2 + n_i^2 + 1} \left( \frac{m_i^2 + n_i^2 + 2}{m_i^2 + n_i^2 + 1} \right) + \tan^{-1} \left( \frac{2m_i n_i \sqrt{m_i^2 + n_i^2 + 1}}{m_i^2 + n_i^2 - m_i^2 n_i^2 + 1} \right) \right],$$

(15b)

for $i = 1, 2, 3, 4, 5$ and,

$$m_1 = \frac{b}{2z}, \quad m_2 = \frac{a + b/2}{2z}, \quad m_3 = \frac{a - b/2}{2z}, \quad m_4 = \frac{2a + b/2}{2z}, \quad m_5 = \frac{2a - b/2}{2z},$$

(15c)

$$n_1 = n_2 = n_3 = n_4 = n_5 = \frac{l}{2z}.$$  

(15d)

Eq. (15) is of such form that is may be used for any tie spacing $a$, tie width $b$, effective tie length $l$, and assumed tie-ballast pressures $p_n$'s.

It should be noted that the AREA Manual for Railway Engineering (1991) recommends a form of the Boussinesq solution, similar to eq. (9), for estimating the vertical stresses in ballast. This formula, as listed in the AREA Manual, is only applicable for estimating vertical stresses directly under an applied concentrated load, not a uniformly loaded area assumed at each tie bottom.
5. VERTICAL STRESS $\sigma_{zz}$ IN BALLAST LAYER ACCORDING TO TALBOT’S FORMULA

The Talbot Committee (1920) developed an empirical formula for vertical stresses within a ballast section. The committee assumed a uniform pressure along the entire length of a tie, $p^*_n$. It is based solely on experimental data obtained from their test setup shown in Fig. 10.

![Fig. 10 Talbot Committee Test Configuration](Talbot, 1920, p.784)

The important characteristics of their test setup are: (1) the rail was supported by either one or three $6'' \times 8'' \times 8'$ wood ties with tie spacing ranging from 18 to 24 inches, (2) various ballast types, such as broken stone, sand, and pebble ballast were used, (3) load cells were placed at various locations and depths within the ballast to measure vertical stresses, and (4) the ties were preloaded five or six times prior to recording the vertical stresses in the test.

During the experiment, no noticeable differences were recorded in the vertical stress distributions for the various ballast types that were used. This independence from the medium material properties agrees with the theory by Boussinesq to calculate vertical stresses.
A best-fit curve was then drawn for the vertical stress data vs. depth and the expression for vertical stresses, $\sigma_{zz}$, was chosen as

$$\sigma_{zz}(z) = \frac{16.8 p_n^*}{z^{5/4}}, \quad (16)$$

where $p_n^*$ is the assumed uniform pressure over the entire length of the tie at the tie-ballast interface, and $z$ is the depth under the tie at which the stress is to be determined. Note that $p_n^*$ is different than $p_n$, which is the magnitude of the assumed pressure distribution only over the outer thirds of the tie, as discussed in Section 3.2.

Also, eq. (16) was chosen from the vertical stresses measurement data under a single loaded tie, not several ties. In order to use the Talbot formula for a series of loaded ties that occur in an actual track, as shown in Fig. 4, the vertical stress contributions from the pressures on adjacent tie bottoms must be somehow included. Therefore, the Talbot Committee used probability theory to generate a more complete formula that approximate the vertical stress in the ballast at any distance $x$ away from the center line under the pressure area. For details, refer to Talbot (1920, p.804) and Bathurst (1997).

Talbot’s formula, eq. (16), is presently listed in the AREA Manual (1991) for estimating vertical stresses within a ballast section. It should be noted that the AREA Manual suggests using the Talbot formula with an effective tie area of $2 \times (b \times L/3)$, instead of the entire tie area $b \times L$ as proposed by the Talbot Committee.

Some additional shortcomings associated with the use of the Talbot formula are: (1) the use of a 6”×8” tie in the tests (whereas today’s standard tie is 7”×9”), (2) the subjective best-fit of their experimental data to generate their empirical formula, (3) the use of ballast that was prepared and compacted in a controlled manner, whereas in practical application, the ballast section may have varying characteristics throughout the cross-section or along any horizontal plane within the section, and (4) the limited range of validity of their empirical formula (note that eq. (16) is not valid for very shallow depths because as $z \to 0$, the predicted vertical stress
under the center of the tie becomes infinitely large). The Talbot Committee acknowledged this when eq. (16) was developed.

For these reasons, it is recommended that the Talbot formula be used with caution, unless a similar experimental program that incorporates modern standards can substantiate the use of Talbot's equation or develop a different empirical formula.

Salem (1966) conducted such an experimental program in order to determine an empirical formula of similar form as the Talbot formula, for estimating the vertical stresses at any point in the ballast due to a single loaded tie. Thus, both formulas require the use of superposition to account for the vertical stress contributions from adjacent ties. However, unlike the Talbot formula, Salem included a correction factor in order for his formula to yield a predicted vertical stress closer to that measured. This correction factor is a function of the magnitude of the load applied to the tie and the depth at which the vertical stress is to be determined. Therefore, this formula cannot be used generally because the correction factors were determined for only specific loadings. Refer to Salem (1966) for details of his empirical formula.

Another empirical formula that approximates the vertical stresses in the ballast was devised by the Japanese National Railway (JNR). It is based a best-fit to their experimental results. It is

\[
\sigma_{zz} (z) = \frac{50 p_n}{10 + z^{1.35}} .
\]  

(17)

Note that \( z \) in eq. (17) is in centimeters. Also in the JNR approach, it was assumed that the pressure is distributed uniformly over the entire tie bottom. For details, refer to the International Railway Congress Association Bulletin (December, 1965). However, note that the JNR test configuration used narrow-gauge tracks and therefore may not be valid for standard-gauge tracks. This formula is also listed in the most recent AREA Manual (1991) for estimating vertical stresses in ballast.
It should be noted that empirical formulas are generally established from data of specific tests and are based on a "best-fit". Therefore, these formulas should be used with caution.

6.0 COMPARISON OF RESULTS OF THE PROPOSED METHOD WITH PUBLISHED TEST DATA

To validate the recommended procedure that is based on the Boussinesq solution, the determined vertical stresses for specific load configurations are compared with test data. The performance of such tests was beyond the scope of this paper. Therefore, the previously published test results by Salem (1966) are being used for comparison purposes.

The Salem experiments generally consisted of an applied load $P^*$ that simulated a single-axle truck, on two rails that rests on a single tie, or a series of three ties. The ties rest on a ballast section supported by a silty, clayey subgrade. Pressure cells were placed throughout the ballast section in order to measure vertical stresses.

Since the ballast was tamped underneath the outer 1/3 of each tie (Salem, 1966, p. 39), it is assumed that the pressure distribution shown in Fig. 3 exists.

For the tests where only a single tie is loaded, the assumed uniform pressure on the bottom outer thirds of the single loaded tie can easily be defined as

$$p_0 = \frac{P^*}{2 \cdot b \cdot l} = \frac{P}{b \cdot l}, \quad (18)$$

where the load carried by one rail $P = P^*/2$, $b = 9''$ and $l = 34''$ for the 7"x9"x8½' ties that were used in the Salem experiments.

Figs. 11 through 13 show the predicted vertical stress distribution curves using the developed solution, eq. (15), and the measured vertical stress values in each test. The measured vertical stresses are shown by a dot (●). Note that for this comparison, in eq. (15a), for a single loaded tie, the pressures under the “non-existing” adjacent ties are $p_{1R} = p_{1L} = p_{2R} = p_{2L} = 0$.

Figs. 11 through 13 show a relatively good agreement between the recorded vertical stresses and those predicted using the proposed method.
**Fig. 11**  Vertical Stress Distribution Under Center Tie Caused By an Applied Load on a Rail Supported by a Single Tie (P = 6,666 lb)

**Fig. 12**  Vertical Stress Distribution Under Center Tie Caused By an Applied Load on a Rail Supported by a Single Tie (P = 10,000 lb)
Fig. 13  Vertical Stress Distribution Under Center Tie Caused By an Applied Load on a Rail Supported by a Single Tie (P = 20,000 lb)

For the tests with three ties supporting the rails, it is assumed that the center tie carries 50% of the load $P$ and that each adjacent tie carries 25% of $P$. Thus, the assumed uniform pressure on each outer 1/3 of the ties are

$$p_0 = \frac{P^*}{4 \cdot b \cdot l} = \frac{P}{2 \cdot b \cdot l} \quad \text{and} \quad p_1 = \frac{P^*}{8 \cdot b \cdot l} = \frac{P}{4 \cdot b \cdot l}.$$  \hspace{1cm} (19)

With $p_0$ and $p_1$ known and using eq. (15), the predicted vertical stress distribution curve for the proposed method are determined. Figs. 14 and 15 show the predicted vertical stress curve using eq. (15) for the shown load configuration. Note that for this comparison, $p_{2R} = p_{2L} = 0$ in eq. (15).

As shown in Figs. 14 and 15, the proposed method again predicts vertical stress values that are reasonable close to the measured values. However, they are generally higher than those measured. This may be attributed to the margin of error associated with the pressure cell readings, and the a priori assumed rail load distribution over the three ties of 25%, 50%, 25% of $P$. 

Fig. 14  Vertical Stress Distribution Under Center Tie Caused By an Applied Load on a Rail Supported by Three Ties ($P = 5,000$ lb)

Fig. 15  Vertical Stress Distribution Under Center Tie Caused By an Applied Load on a Rail Supported by Three Ties ($P = 10,000$ lb)
The shown comparisons indicate that the proposed method yields reasonably close results. It is therefore suggested that it is suitable for predicting the vertical stress in ballast for design analyses.

In conclusion, it should be noted that this method does not take into account the weight of ballast or of the rail-tie structure. However, they may be added using superposition. It may be shown that these additional vertical stresses are negligible compared to those caused by the wheel loads of cars and locomotives.

7. RECOMMENDED METHOD FOR THE DETERMINATION OF $\sigma_{zz}$

It can be shown how the extensive calculations associated with the Boussinesq method, may be avoided by producing design charts for the vertical stresses at any depth within the ballast section, below the point of maximum pressure under the rail.

For the construction of the design charts, various load configurations of a typical freight consist comprised of a locomotive on 3-axle trucks and cars on 2-axle trucks are considered. Of all the wheel loads associated with a train, the load configuration that consists of two 2-axle trucks will be used for the analysis. This load arrangement represents the adjacent trucks of two adjoining cars. Typical wheel spacings are used. The two 2-axle trucks are separated by a distance of 80.5″, where the two axles of each freight truck are 70″ apart. Note that for typical $k$-values, the actual maximum rail pressure may occur under the rear truck of the last car of the train. However, the difference between the maximum pressures at these locations is very small. Therefore, for the design of an adequate ballast layer thickness, the use of the general load configuration of two adjoining cars, as shown in Fig. 16, is sufficient.

For specified track parameters [track modulus $k$, center-to-center tie spacing $a$, tie dimensions $b, l$, rail properties $E, I$, and wheel load $P$], the total pressure distribution $p(x)$ under the rail can be produced by superposing $p(x)$-curves caused by each wheel load at their respective location along the rail. An example of the total pressure distribution curve (under the rail) for the critical load configuration is shown in Fig. 17.
Fig. 16  General Load Configuration for Determining Largest Vertical Stress (Under Axles of Two Adjoining Cars)

Fig. 17  Superposition of the Pressure Distribution that the Rail Exerts on the Ties
Once this total pressure distribution curve has been constructed, the maximum ordinate of that curve is located. Then, the maximum pressure value is used with eqs. (7) and (8) to determine the assumed uniform pressure under each outer 1/3 of the center tie, $p_0$, under which the maximum $p(x)$-value is located. The $p_{1R}$-value is calculated by first determining the pressure ordinate at a distance $a$ to the right of the maximum value, and then using eqs. (7) and (8) to determine $p_{1R}$, as was previously done to determine $p_0$. Similarly, this procedure can be performed to calculate $p_{1L}, p_{2R}, p_{2L}, \text{etc.}$ Once all of the $p_n$-values are determined, eq. (15) can be used to determine the vertical stress vs. depth distribution below the location of maximum pressure under the rail.

This procedure was performed for $k = 3,000 \text{ lb/in}^2$, 132RE rail ($E = 30 \times 10^6 \text{ lb/in}^2$, $I = 87.9 \text{ in}^4$), and a center-to-center tie spacing $a = 20 \text{ in}$. A series of vertical stress vs. ballast depth curves, shown in Fig. 18, have been plotted that account for the vertical stress contributions of all

![Fig. 18 Boussinesq Vertical Stress Distribution Curves for Various Maximum Tie-Ballast Pressures $p_0$ Caused by the General Load Configuration with $k = 3,000 \text{ lb/in}^2$ and $a = 20 \text{ in}$]
five loaded ties (i.e., the center tie and the two adjacent ties to either side) caused by various magnitudes of the wheel loads of the two adjoining cars shown in Fig. 16.

To utilize this design chart, only the maximum tie-ballast pressure $p_0$ has to be known. With this value, the corresponding curve on Fig. 18 is located (if the $p_0$-value is located between curves, interpolate between neighboring curves). Because these curves already include the vertical stress contributions from the tie-ballast pressures from adjacent ties, the $p_{1R}, p_{1L}, p_{2R}$, and $p_{2L}$-values do not have to be determined. The use of this design chart, shown in Fig. 18, is intended to eliminate the long calculations associated with the Boussinesq method and to generate a quick, accurate prediction of vertical stresses in ballast at any specified depth.

It should be noted that the magnitudes of the vertical stress contribution from adjacent ties will vary with different tie spacing $a$, and track modulus, $k$. For these cases, another design chart must be constructed. Design charts, similar to Fig. 18, were constructed for $k = 3,000$ lb/in$^2$ and $a = 18$ in, and $k = 6,000$ lb/in$^2$ and $a = 24$ in (i.e., well-maintained concrete-tie track) and are shown as Figs. 19 and 20, respectively.

It was found that these curves do not significantly differ for various rail types (i.e. 115RE, 119RE or 132RE). Therefore, Figs. 18 through 20 can be generally used for the design of an adequate ballast section for a prescribed $k$, tie spacing $a$, and any rail type.
Fig. 19 Boussinesq Vertical Stress Distribution Curves for Various Maximum Tie-Ballast Pressures $p_0$ Caused by the General Load Configuration with $k = 3,000$ lb/in$^2$ and $a = 18$ in.

Fig. 20 Boussinesq Vertical Stress Distribution Curves for Various Maximum Tie-Ballast Pressures $p_0$ Caused by the General Load Configuration with $k = 6,000$ lb/in$^2$ and $a = 24$ in (typical values for well-maintained concrete-tie tracks)
8. APPLICATION OF THE PRESENTED DESIGN CHARTS FOR THE DETERMINATION OF NECESSARY BALLAST DEPTH

To illustrate how these design charts are to be used in railway engineering, consider two 100-ton cars on a well-maintained wood tie track (i.e. track modulus $k = 3,000$ lb/in²) with 136RE rail. It is assumed that a wheel load of a 100-ton car is 32,500 lb. The wood ties have a 9”×7” cross-section and are 8’ 6” long. Consequently, the effective tie length, at each end, $l = L/3 = 34”$.

Using the analytical approach discussed previously, a total pressure distribution $p(x)$ under the rail is constructed by superposing the $p(x)$ curves caused by each wheel load at their respective location along the rail, as shown in Fig. 17. Each pressure distribution curve was multiplied by 1.5 to account for the well-documented nonlinear response of the ballast. Refer to Kerr and Shenton (1986, p.1120) for details.

As shown in Fig. 17, the maximum pressure of 1,173 lb/in occurs at approximately 6 inches from the inside edge of the outer wheel of either truck as shown in Fig. 16. It is at this maximum that the critical vertical stress distribution through the ballast has to be determined. Using eq. (7), the maximum rail-seat force is easily determined as

$$F_0 \equiv p(0) \cdot a \quad \rightarrow \quad F_0 \equiv (1173.5 \text{ lb/in}) \cdot (20 \text{ in}) = 23,470 \text{ lb}, \quad (20)$$

and, using eq. (9), the tie-ballast pressure $p_0$ equals

$$p_0 = \frac{F_0}{b \cdot l} \quad \rightarrow \quad p_0 = \frac{23,470 \text{ lb}}{(9 \text{ in}) \cdot (34 \text{ in})} = 76.7 \text{ lb/in}^2. \quad (21)$$

With the maximum tie-ballast pressure $p_0$ determined, Fig. 18 can now be utilized. Interpolating between the $p_0 = 75 \text{ lb/in}^2$ and $p_0 = 80 \text{ lb/in}^2$ curves, the depth at which the vertical stress meets the AREA criterion for allowable vertical stress on subgrade ($\sigma_{zz} = 20 \text{ lb/in}^2$) can be estimated. For the present example, the required depth is estimated to be 30 inches, as shown in Fig. 21.
For comparison purposes, the required ballast depth is calculated according to the first two recommended formulas listed in the AREA Manual (1991). The first formula by Talbot (1920) is

$$\sigma_z(z) = \frac{16.8 \, p_n}{z^{5/4}}. \tag{16}$$

The $p_n$-term in eq. (26) is the assumed uniform pressure over the effective tie area, as calculated by eq. (8). Note, however, that this $p_n$-term is not the assumed uniform pressure over the entire tie base, as given by Talbot in eq. (16). Using eq. (26) and $p_n = p_0 = 76.7 \, \text{lb/in}^2$, the vertical stress distribution under the location of maximum pressure is shown in Fig. 22. As shown, the required ballast depth to satisfy the AREA criterion is about 28 inches.

Fig. 21 Use of Boussinesq Vertical Stress Distribution Curves for Various Maximum Tie-Ballast Pressures ($p_0$) Caused by the Critical Load Configuration
The second formula listed in the AREA Manual is the JNR formula, eq. (17). Note that this formula requires the use of centimeters for \( z \). The vertical stress distribution using eq. (17) and \( p_n = p_0 = 76.7 \text{ lb/in}^2 \) is also shown in Fig. 22. As shown, the required ballast depth to satisfy the AREA criterion is approximately 19 inches (a much smaller depth than previously calculated).

As shown in Fig. 22, the required ballast depths calculated by the recommended Boussinesq method and the Talbot formula, as listed in the AREA Manual, are relatively close. However, as detailed in Section 5, the Talbot formula was originally given for an assumed...
pressure distribution over the entire length of the tie and does not allow the vertical stress contribution from adjacent ties to be calculated. The AREA Manual recommends using the Talbot formula with an assumed pressure distribution over the outer thirds. Therefore, it appears that the Talbot formula as recommended in the AREA Manual is used incorrectly. The fact that the required ballast depth calculated from the Talbot formula, as listed in the AREA Manual, is close to that calculated using the recommended Boussinesq method is coincidental. Refer to Talbot (1920) and Bathurst (1997) for details.

Due to the reasonably good accuracy between the predicted vertical stresses and various test data at any depth within the ballast section shown in Section 8, it is believed that the Boussinesq method and the presented design charts can provide a quick and relatively accurate approximation of vertical stresses in the ballast for the determination of the required ballast depth that satisfies the maximum allowable subgrade criterion recommended by AREA.
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