Improved Spiral Geometry for High Speed Rail

by

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Abstract

An improved method for the design of spirals for passenger railroads is explained and illustrated.

The new method is based on three principles as follows:

- The traditional principle that track curvature should correspond to vehicle roll motion and should change as the roll changes so that passengers always experience zero (or small) lateral acceleration.

- A principle proposed here that roll motion should be controlled so that roll velocity, roll acceleration, and rate of change of roll acceleration (roll jerk) all stay within comfortable limits.

- Another principle proposed here that roll motion should be about an axis at about chest height for standing or seated passengers so that passengers experience minimum lateral acceleration due to roll per se.

The new method is mathematically a little more complex than the traditional method based on ordinary spirals but is easily applied with the help of a computer and gives much better results.

The new method is illustrated by applications to a restrictive pair of reverse curves on the Northeast Corridor and to a typical isolated curve. The applications show that the new method will permit beneficial increases in superelevation for some existing reverse curves that cannot be adequately elevated based on traditional criteria. The applications also show that existing curves could be redesigned with large superelevations without deviating significantly from present alignments and thus without requiring any additional right-of-way.
1. Introduction

This paper presents and illustrates a new method for design of spiral transition curves for railroad tracks. The new method of spiral design is like the traditional method in the way that it uses super-elevation to counteract the centripetal force that is present during curve traversal. The difference between the new method and the traditional method is in what they attempt. The traditional method starts by looking at the plan view of the track alignment as a geometrical shape and seeks to make that shape “smooth.” The method proposed here starts by looking at the vehicle roll motion during spiral traversal and seeks to make that motion “smooth”. The proposed method is superior to the traditional method both in concept and in practical performance. The present proposal can be compared to that in a good paper by G. Presle and H. L. Hasslinger (ref. 8) which documents the poor performance of traditional spiral design and also documents substantial improvement that can be realized if the traditional spiral alignment and superelevation profile are both adjusted so that they connect more “smoothly” at each end.

The new method is superior in general, and primary focus should be placed on the noticeable comfort improvement that it can make in curve transition dynamics for all passenger train operations. However, mention can also be made of two situations where the new method offers distinctive benefits. First, some reverse curves are subject to speed restrictions because the two curves of the reverse curve pair are so close to one another that the traditional method of spiral design does not allow as much superelevation as would be desired. For such reverse curve pairs the new method will allow increased superelevation that will eliminate or lessen the speed restrictions. Second, there is sometimes a desire to take an existing freight rail route and upgrade it for high-speed passenger service. Existing spirals on such routes are typically short. Ability to lengthen the spirals and thereby allow superelevations appropriate for passenger service is likely to be hampered by property boundaries. In such cases the proposed spiral design method will allow superelevations that achieve or are closer to the desired values. The examples presented herein illustrate both of these distinctive benefits.
The impetus for development of the new spiral design method came from an exploration (ref. 1) of the extent to which high superelevations could allow trip times between New York and Washington to be reduced. In that context, the new method shows that curve alignments including spirals associated with high superelevations can be laid out within existing right-of-way property lines.

The new method has the disadvantage that derivation of working formula is more complex than for the traditional simple spirals. However, the ideas involved are all elementary, and once appropriate programs have been written, the calculations needed to develop spirals of the proposed type for particular curves can be done very quickly on a computer.

Readers who are not versed in track design may find it helpful to begin by with the figures in section 11 near the end of the paper. Readers who are not interested in mathematics will want to skip over sections 4 through 6.

2. PHYSICAL CONCEPTS

The proposed new method of spiral design is based on three physical ideas, as follows.

2.1 Balance of Lateral Force

Ideally, the track curvature (as seen from above) should be coordinated with the track superelevation so that when a train traverses a curve at design speed, the effect of the curvature and superelevation just balance one another so that curve traversal does not subject passengers to any steady lateral acceleration. The mathematical expression of this coordination is easily written down as follows.
Let:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>length measured along track</td>
</tr>
<tr>
<td>r</td>
<td>roll of vehicle.</td>
</tr>
<tr>
<td>b</td>
<td>bearing angle between direction of track (as seen from above) and a reference direction, e.g., east.</td>
</tr>
<tr>
<td>v</td>
<td>vehicle speed</td>
</tr>
<tr>
<td>g</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>c</td>
<td>derivative of b with respect to track length.</td>
</tr>
</tbody>
</table>

(In this paper r will actually refer to the angle of super-elevation of the track, as that is what can be controlled.)

This is the curvature of track (as seen from above) and = 1/(local radius of curvature), when track not straight.

The component of gravity felt by a passenger as a transverse acceleration (i.e. in the plane of the floor of the car) is

\[ g \sin(r) \]

(The * symbol is used to indicate multiplication.) The centripetal acceleration of the vehicle and passengers due to curve negotiation is

\[ c \times v^2 \]

The component of centripetal acceleration in the plane of the floor of the car is

\[ c \times v^2 \times \cos(r). \]

Requiring that the components of gravitational and centripetal acceleration in the plane of the car floor just balance we have the constraint that

\[ c \times v^2 \times \cos(r) = g \times \sin(r). \]

or, assuming that roll angle is known as a function \( r(s) \) of distance \( s \) along the track, the relation determining track geometry in the transition is

\[ \frac{db}{ds} = \frac{g}{v^2} \times \tan(r(s)) \quad (1) \]

so that \( b(s) \) can be found by integrating \( \tan r(s) \). Once \( b(s) \) is known, the coordinates of points along the curve are obtained by the further integrations,

\[ x(s) = x(0) + \text{integral} \ (0 \ to \ s) \ dt \times \cos(b(t)) \quad (2) \]
\[ y(s) = y(0) + \text{integral} \ (0 \ to \ s) \ dt \times \sin(b(t)). \quad (3) \]

The relationship between curvature and roll angle could be interpreted as a way to determine roll angle as a function of location in a spiral if curvature versus location were already prescribed. However, this study
adopts the point of view that roll angle should be controlled subject to constraints appropriate to roll motion per se and uses the above relationship to determine spiral geometry (as seen from above) after roll motion has been determined.

2.2 Control of Vehicle Roll Angle

The second idea is that angle of vehicle roll during spiral traversal can and should be controlled for comfort in somewhat the same manner that vehicle longitudinal motion is customarily controlled. The basic ingredients of comfortable linear motion control are:

- a limit on time rate of change of acceleration
  (often called the acceleration ramp rate or jerk)
- a limit on magnitude of acceleration
- a desired distance of travel

In the case of the roll degree of freedom the corresponding ingredients are:

- a limit on roll jerk
- a limit on roll acceleration
- a limit on roll velocity
- a desired change in roll angle (from that in one section of constant curvature to that in the next section of constant curvature).

Roll velocity is added to the list for two reasons. First, some passengers may be sensitive to it via sensory mechanisms related to balance and motion sickness. Second, the roll velocity of the vehicle is tied to the rate of change of rail superelevation with distance, sometimes called the warp of the track, namely

\[ \frac{dr}{dt} = v \times \frac{dr}{ds}. \]

The warp of the track causes the track at one end of a vehicle to be rolled relative to that at the other end. If this relative roll were too large, then, with a conventional suspension there would be degradation in the ability of the suspension to filter the effects of rail profile irregularities and in its ability to resist
derailment. (Velocity of linear motion is usually subject to limits also, but that is because of factors such as guideway irregularity and energy consumption, not because of any intrinsic discomfort of linear velocity.)

Desired change of roll angle is typically from zero (i.e. vertical) on straight track to the value that will balance (some or all of) the centripetal acceleration of a constant radius curve.

Figure 1 illustrates roll motion (of the track) which conforms to the above 4 limits, and which has roll velocity and acceleration both zero at each end. If the transition is followed by a segment of track with constant superelevation and constant curvature, then with this motion there is no roll acceleration or roll jerk other than that which is explicitly present in the transition. The spirals presented herein are based on roll motion that conforms to this pattern.
In relation to the roll motion model of Figure 1, it is convenient to introduce the symbols $J$, $A$, $W$, and $R$ respectively for the maximum values of roll jerk, roll acceleration, roll velocity, and roll angle. It is also convenient to introduce the symbols $s_1$, $s_2$, $s_3$, and $s_4$ respectively for the arc lengths from the symmetry center of the spiral to the 4 points on the right side where the value of roll jerk switches from one constant value to another.

Since each of the lower three curves of Figure 1 is the integral of the curve above it, the following relationships among the above parameters may be written down by inspection.

\[
\begin{align*}
    s_2 - s_1 &= s_4 - s_3 = \frac{A}{J} \\ 
    s_3 - s_1 &= \frac{W}{A} \\ 
    s_4 + s_1 &= \frac{R}{W}
\end{align*}
\]
From (4) & (5) we have

\[ s_4 - s_1 = W/A + A/J \]  

(7)

and using (6) & (7) we obtain

\[ s_1 = 0.5 \times (R/W - W/A - A/J) \]  

(8)

and

\[ s_4 = 0.5 \times (R/W + W/A + A/J) \]  

(9)

Another form of transition curve can be obtained by omitting the region of constant curvature between spirals and letting the transition as a whole consist of two spirals back-to-back. In this case, there is no need to have roll acceleration = 0 where the two spirals meet. Roll motion for back-to-back spirals of this type that just conforms to the four limits is illustrated in Figure 2. A curve of this type will be as short as possible consistent with the stated limits and could be useful in some situations.

**Figure 2. Specialized roll motion for continuously spiraling transition between two tangents**

In relation to Figure 2 it is useful to introduce symbols like those related to Figure 1 but with arc length values \( s_1 \) through \( s_6 \) corresponding to the 6 points to the right of the symmetry center where the roll jerk changes value. Expressions for boundary points in Figure 2 may then be obtained, in some case by a little algebra (and from here on multiplication will sometimes be indicated by simple juxtaposition with out the * symbol), as
\[ s_1 = W/A - A/( 2 J ) \]  
\[ s_2 = W/A + A/( 2 J ) \]  
\[ s_3 = R/W - A/( 2 J ) + A^3/( 24 W J^2 ) \]  
\[ s_4 = R/W + A/( 2 J ) + A^3/( 24 W J^2 ) \]  
\[ s_5 = R/W + W/A - A/( 2 J ) + A^3/( 24 W J^2 ) \]  
\[ s_6 = R/W + W/A + A/( 2 J ) + A^3/( 24 W J^2 ) \]

It may be helpful conceptually to look at the roll motion of Figure 1 as a motion which starts at one constant value of roll angle and changes gracefully in the simplest and most direct way to another constant value. One feature of the simplicity is that throughout the transition, the roll velocity is always \( \geq 0.0 \) or else \( \leq 0.0 \). The associated track curvature will therefore increase or decrease monotonically.

In contrast to that, the roll motion of Figure 2 can be regarded as the simplest graceful roll motion in which roll velocity is first \( \geq 0.0 \) and then \( \leq 0.0 \) (or vice versa). The associated track curvature will increase and then decrease (or vice versa), and the resulting transition can be called a double spiral. When the magnitude of the increase and decrease are the same, the double spiral will connect two tangents or two curves of identical curvature. It can easily be generalized to connect a tangent and a curve or two curves of differing curvature by having the magnitude of the roll increase and roll decrease differ. It has been suggested by Jan Zicha (ref. 7) that the spiral design method presented here might allow design of main line passenger turnouts with improved characteristics. The back-to-back spiral of Figure 2 would be appropriate for that application.

The roll motion illustrated in Figures 1 and 2 is mathematically simple (a polynomial of 1st, 2nd, or 3rd degree) within each of the regions of constant or zero jerk. The function which describes the motion as a whole embodies the complication of tests to find which zone a given point is in, followed by evaluation of the polynomial applicable to that zone. The calculus of variations can be used to obtain a differential equation from whose solutions one can construct functions that come reasonably close to satisfying the
same physical requirements as the motions illustrated in Figures 1 and 2. Such functions would have the advantage of having continuous derivatives throughout the transitions. Solutions of that type could be of interest if they offered better dynamic performance or if they lead to simpler computation. However, solutions of that type appear to provide inferior performance and to involve more complex calculations. They will therefore not be considered here.

2.3 Height of Vehicle Roll Axis

Roll motion of a real vehicle can be viewed as consisting entirely of rotation about some general roll axis or equivalently as consisting of a combination of rotation about the vehicle’s characteristic longitudinal axis (one of the three principal axes of the inertia ellipse) plus some vertical and lateral motion of the characteristic axis.

In order to minimize forces applied to a vehicle, it would be desirable to have the roll axis coincided with the vehicle’s characteristic axis. In order to minimize torque about a longitudinal axis imposed on the vehicle’s suspension, it would be desirable to have the roll axis pass through the so-called center of percussion with respect to transverse force applied to the wheel treads. In order to minimize transverse accelerations imposed on passengers, it would be desirable to have the roll axis at about chest height of standing or seated passengers (say, 7.0 feet above the rail).

While working on the last example in Section 11.1, the author found that for a spiral between existing reverse curves, increase of the roll axis height causes an increase in the length of the spiral and a reduction of the maximum rate of change of superelevation with distance (i.e., the maximum warp, which is discussed in Sections 6 & 7). That example is one in which the existing reverse curves are offset from one another by a distance which is less than the offset distance which would be natural in a corresponding
new design for significantly higher speed. It turns out that artificial increase of the roll axis height can be used to compensate for deficiency of the offset. Naturally, if roll axis height is made larger than 7 ft., passengers will experience some lateral acceleration as a result of the roll acceleration about the roll center.

This paper presents transition geometry examples based on roll axis heights of 0.0 ft., 7.0 ft., and 14 ft. We expect that an axis height close to 7 ft. will be most satisfactory in practice except in cases of redesign of existing reverse curves whose offsets are less than would be desired.

Conceptually, the spiral design method proposed here has four basic elements. A), it models the vehicle as a point mass at the height of the roll axis above the guiding rails. B), it determines the roll of the rails versus arc length per section 2.2. C), it determines the curvature (as seen from above) of the path of the point mass versus roll per section 2.1, from the curvature it determines the path of the point mass. D), and it then determines the path of the rails based on roll axis height and on roll angle versus arc length.

3. SCOPE OF PAPER

With the foregoing as background, the general scope of this paper can be laid out as follows.

1. We assume that within regions of track where curvature is constant, vehicle roll angle is constant such that passengers feel no (or little) lateral acceleration.

2. Between two regions of constant curvature, the vehicle roll angle is to change from the value proper in the earlier region to the value proper in the later region. We want to develop roll motions that achieve the required change of roll angle efficiently without exceeding limits corresponding to thresholds of passenger sensitivity and/or limits for good suspension performance.
3. We will establish track alignment in the transition region by the requirement that curvature at each point on the trajectory followed by the vehicle roll center be based on the roll angle established for that point such that passengers continue to feel no (or little) lateral acceleration at design speed.

4. We want to be able to plot the resulting alignments, to see what an alignment looks like when the roll angle change is made as rapidly as possible (subject to roll motion constraints), and to see how the alignment changes when the value of jerk is reduced from the limiting value.

5. We want to see how alignments constructed as outlined above compare with some typical existing track alignments. We would like specifically to see if curves in the Amtrak route between Washington and New York can be redesigned with greatly increased superelevation to accommodate much higher speeds and yet stay within existing railroad property lines. We also wish to see if some existing reverse curves that cause speed restrictions because of inadequate superelevation can be redesigned with increased superelevation so that the speed restrictions are eliminated or lessened.

4. ALGEBRA FOR ROLL ANGLE VS. ARC LENGTH

This section presents expressions for $r(s)$ (roll angle versus length along track) corresponding to Figures 1 and 2.

In Figure 1, each curve other than roll angle has symmetry (or anti-symmetry) about the middle value of distance, and roll angle minus its average value also has such symmetry. It is therefore convenient for
Figure 1 to place the distance origin at the midpoint of the Figure. The expressions for \( r(s) \) corresponding to the normal transition of Figure 1 are given in Table 1.

The curves in Figure 2 all have symmetry about the midpoint, and the are treated analogously. The corresponding expressions are given in Table 2. The same formulae can be used for asymmetric double spirals that can connect curves of differing radii if different parameters are used on opposite sides of the point where roll velocity is zero.

**Table 1. Formulae for roll angle vs. arc length for spiral with monotonic curvature corresponding to Figure 1**

<table>
<thead>
<tr>
<th>arc length in zone</th>
<th>roll acceleration</th>
<th>roll velocity</th>
<th>roll angle *</th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td>to</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in zone</td>
<td>0</td>
<td>( W )</td>
<td>( W_s )</td>
</tr>
<tr>
<td>at ( s_1 )</td>
<td>0</td>
<td>( W )</td>
<td>( r_1 = R/2 - (W/2)(W/A + A/J) )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( s_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in zone</td>
<td>- ( \text{sgn}(s)J(</td>
<td>s</td>
<td>- s_1) )</td>
</tr>
<tr>
<td>at ( s_2 )</td>
<td>- ( A )</td>
<td>( w_2 = W - A^2/(2J) )</td>
<td>( r_2 = R/2 - A^2/(6J^2) - (W/2)(W/A - A/J) )</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( s_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in zone</td>
<td>- ( \text{sgn}(s)A )</td>
<td>( W_2 - A(</td>
<td>s</td>
</tr>
<tr>
<td>at ( s_3 )</td>
<td>- ( A )</td>
<td>( w_3 = A^2/(2J) )</td>
<td>( r_3 = R/2 - A^2/(6J^2) )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( s_4 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in zone</td>
<td>- ( \text{sgn}(s)[A - J(</td>
<td>s</td>
<td>- s_3)] )</td>
</tr>
<tr>
<td>at ( s_4 )</td>
<td>0</td>
<td>0</td>
<td>( R/2 )</td>
</tr>
</tbody>
</table>

* roll angle, \( r \), is shown normalized for a symmetric reverse curve. \( r(s) \) for a spiral from a tangent at \( s = -s_4 \) to a curve at \( s = s_4 \) is obtained by adding \( R/2 \) to each of the above expressions for \( r \). The function \( \text{sgn}(x) \) sometimes called the sign of \( x \) or the signum function is defined as \( x/|x| \) for \( x <> 0 \) and 0 for \( x = 0 \).
<table>
<thead>
<tr>
<th>arc length in zone</th>
<th>roll acceleration</th>
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<th>roll angle *</th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td>to</td>
<td>at</td>
<td>in zone</td>
</tr>
<tr>
<td>0</td>
<td>s₁</td>
<td></td>
<td>- A</td>
</tr>
<tr>
<td>s₁</td>
<td>s₂</td>
<td></td>
<td>- A + J(</td>
</tr>
<tr>
<td>s₂</td>
<td>s₃</td>
<td></td>
<td>0</td>
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<tr>
<td>s₃</td>
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<td>J(</td>
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<tr>
<td>s₄</td>
<td>s₅</td>
<td></td>
<td>A</td>
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</tbody>
</table>
5. INTEGRATIONS TO OBTAIN GEOMETRY OF SPIRAL

The remainder of the present paper focuses on development and application of monotonic spirals based on the model of Figure 1. The present paper does not go any farther with double spirals based on the model of Figure 2.

Having obtained expressions for \( r(s) \), the roll angle (or superelevation) vs. arc length (length along the track), we next need to integrate equation (1),

\[
\frac{db}{ds} = \left(\frac{g}{v^2}\right) \tan(r(s))
\]

where \( b(s) \) is the compass bearing (in radians) along the track at location \( s \) and \( db/ds \) is the curvature. After \( b(s) \) is available we will need to integrate equations (2) and (3) namely,

\[
\begin{align*}
x(s) &= x(0) + \int_0^s dt \cos(b(t)) \\
y(s) &= y(0) + \int_0^s dt \sin(b(t)).
\end{align*}
\]

In the regions where \( r(s) \) is linear in \( s \), say \( r(s) = a + c \cdot s \), equation (1) has the closed form solution

\[
b(s_2) = b(s_1) - \left(\frac{g}{v^2 \cdot c}\right) \ln \left[\frac{\cos(a + c \cdot s_2)}{\cos(a + c \cdot s_1)}\right].
\]

However, in the regions where \( r(s) \) is quadratic or cubic, a solution in terms of elementary functions is not available. Moreover, solutions of equations (2) and (3) will not be available in terms of known functions even when \( r(s) \) is linear in \( s \).

Therefore, for simplicity of programming, we have used ordinary numerical integration for both integrations throughout the entire range of each one. The results are accurate and run times on a contemporary personal computer are about one second.

6. PARAMETERIZATION OF SPIRAL

We turn now to the question of how to choose the constants of Figure 1, which determine the actual geometry of a transition defined according to Table 1. As introduced, these constants are

\[
\begin{align*}
R &- \text{ change in vehicle roll (track superelevation)} \\
W &- \text{ maximum roll velocity, which when expressed as change in roll angle per unit distance along track, is the maximum warp of the track.} \\
A &- \text{ maximum value of change in warp per unit length along track.} \\
J &- \text{ change in } A \text{ per unit length (proportional to the roll acceleration ramp rate, i.e. the rate at which roll acceleration changes)}
\end{align*}
\]

We assume that vehicle speed is constant during spiral traversal so that derivatives with respect to time are proportional to corresponding derivatives with respect to distance along the track.

It appears convenient to choose two of the above quantities directly, namely,
\begin{itemize}
  \item \textbf{R} - is the difference between the constant values of roll angle appropriate after and before the transition, and
  \item \textbf{J} - represents a roll acceleration ramp rate (jerk) and provides a direct control over the compromise between gentleness and efficiency of the transition.
\end{itemize}

Rather than choosing the other two constants, \textbf{W} and \textbf{A}, directly, we prefer to introduce two auxiliary variables,

\begin{itemize}
  \item \textbf{f} = \text{fraction of roll acceleration distance in which the roll acceleration is constant, and}
  \item \textbf{h} = \text{fraction of total transition distance in which the roll velocity, \textbf{W}, (which is proportional to track warp) is constant.}
\end{itemize}

The motives for introducing these parameters are

- Their use eliminates the possibility of choosing \textbf{W} and \textbf{A} values which do not yield the type of motion shown in Figure 1,
- They provide an intuitive way to visualize how the transition is developed consistent with the constants \textbf{R} and \textbf{J}.

Algebraically, introducing the capital \textbf{S} symbols,

\begin{align*}
  S_1 &= \frac{R}{W} \\
  S_2 &= \frac{W}{A} \\
  S_3 &= \frac{A}{J}
\end{align*}

and noting the expressions given in Section 2.2 for the little \textit{s} distances, the definition of \textbf{f} takes the form

\begin{equation}
S_2 - S_3 = f \times (S_2 + S_3)
\end{equation}

so that

\begin{equation}
S_3 = \left[ \frac{(1 - f)}{(1 + f)} \right] \times S_2.
\end{equation}

The definition of \textbf{h} takes the form

\begin{equation}
S_1 - S_2 - S_3 = h \times (S_1 + S_2 + S_3)
\end{equation}

so that

\begin{equation}
S_2 + S_3 = \left[ \frac{(1 - h)}{(1 + h)} \right] \times S_1.
\end{equation}

Further, with \textbf{R} and \textbf{J} given, we note

\begin{equation}
S_1 \times S_2 \times S_3 = \frac{R}{J}.
\end{equation}

We introduce the abbreviations
F = \(\frac{1 - f}{1 + f}\), \hspace{1cm} (i)

F_1 = 1 + F,

and

H = \(\frac{1 - h}{1 + h}\). \hspace{1cm} (j)

From (e) we can write

\[ S_2 + S_3 = (1 + F) \cdot S_2 = F_1 \cdot S_2. \] \hspace{1cm} (k)

From (g) we then write

\[ F_1 \cdot S_2 = H \cdot S_1 \]

or

\[ S_2 = \frac{H}{F_1} \cdot S_1. \]

Then, using (e) again

\[ S_3 = (F \cdot H/F_1) \cdot S_1, \]

and (h) becomes

\[ \left( F \cdot \frac{H^2}{F_1^2} \right) \cdot S_1^3 = \frac{R}{J}, \]

so that

\[ S_1 = \left[ \frac{F_1}{H} \right]^{2/3} \cdot \left( \frac{1}{F} \right)^{1/3} \cdot \left( \frac{R}{J} \right)^{1/3}, \]

\[ S_2 = \left[ \frac{H}{\left( F \cdot F_1 \right)} \right]^{1/3} \cdot \left( \frac{R}{J} \right)^{1/3}, \]

and

\[ S_3 = F^{2/3} \cdot \left[ \frac{H}{F_1} \right]^{1/3} \cdot \left( \frac{R}{J} \right)^{1/3}. \]

Then

\[ W = \frac{R}{S_1} = \left[ \frac{H \cdot R}{F_1} \right]^{2/3} \cdot \left( \frac{F}{J} \right)^{1/3} \]

and

\[ A = \frac{W}{S_2} = \left[ \frac{H \cdot R}{F_1} \right]^{1/3} \cdot \left( \frac{F \cdot J}{2} \right)^{2/3}. \]

Another relation that may be derived is

\[ \text{total spiral length traversed by roll axis} = 4 \cdot S_3/\left( \frac{1 - f}{1 - h} \right). \]
Since $S_3$ is the length in which acceleration ramps up or down, this expression can easily be seen to be correct in the special case that $f$ and $h$ are both zero (0).

7. PHYSICAL VALUES OF PARAMETERS

**Vehicle roll**, $r$, for a vehicle traveling at speed $v$ in a curve with constant radius, $\text{Radius}$, is given by equation (1) in the form

\[
\frac{1}{\text{Radius}} = \left( \frac{g}{v^2} \right) \tan r
\]

or

\[
r = \arctan \left( \frac{v^2}{g \times \text{Radius}} \right).
\]

The values that we will use as general examples are given in Table 4 in section 9. Pursuant to the author's previous exploration of use of greatly increased superelevation to achieve more competitive trip times between New York and Washington (ref. 1), Table 4 in Section 9 shows some examples with superelevations up to about 45 degrees. With vehicle roll of 45 degrees and curve traversal at the corresponding balancing speed, passengers will experience a 40% increase in apparent body weight, and vehicle suspensions will need to be designed for large and somewhat rapid variation of apparent load. Handling that degree of weight increase would require that passengers be seated. We would not suggest going above that value even with passengers seated.

**Track warp**, $W$, has been limited in traditional railroad practice to values expressed in terms of superelevation change as shown in table 3 (which is taken from a Penn Central Co. Track Maintenance Manual, ref 2).

<table>
<thead>
<tr>
<th>authorized speed (mph)</th>
<th>change in superelevation per 31 ft. along track</th>
</tr>
</thead>
<tbody>
<tr>
<td>up to 50</td>
<td>0.5 inch</td>
</tr>
<tr>
<td>51 to 70</td>
<td>0.375 inch</td>
</tr>
<tr>
<td>over 70</td>
<td>0.25 inch</td>
</tr>
</tbody>
</table>

We assume that the values for speeds above 50 mph were adopted to reduce lurching that would otherwise be noticeable because of the form of the traditional railroad spiral. (The traditional spiral is conceptually a plain Fresnel spiral with track warp changing abruptly between zero and its full value at
each end of the spiral or at points somewhat beyond the ends of the spiral). We assume that with a 60 inch gauge (rail center to rail center), a superelevation change rate of 2.5 inch per 60 ft will be acceptable for suspension performance and for running stability as long as roll angular acceleration and jerk are well controlled and track surface geometry is accurate. The corresponding limit placed on track warp is 7 E-4 radians/ft. With respect to passenger comfort, at 210 mph this represents roll through 45 degrees in about 3.6 seconds, which we presume to be acceptable, but also about as high a rate as would be readily accepted.

For **roll acceleration**, we are not aware of any existing criteria from a passenger comfort point of view. In order to have a value for use in the illustrations, we will make a rough estimate as follows. The Budd Silverliner cars which operate around Philadelphia have a natural frequency of roll oscillation of about 0.8 Hz and while traversing some turnouts exhibit roll oscillations with an amplitude which the author judges to have been approximately

\[
\text{peak airbag compression from neutral} = 1.0'' = 0.02 \text{ radians} \\
0.5 \times \text{distance between air bags} = 50.0''
\]

The corresponding peak roll acceleration would be

\[
(2 \times 3.14 \times 0.8)^2 \times 0.02 = 0.5 \text{ rad/sec}^2
\]

In the author’s judgment, this roll acceleration is not at all uncomfortable. (The oscillatory roll motion just described did produce some discomfort, but that was due to an accompanying lateral acceleration rather than to the roll per se). Therefore, for illustrative purposes, we will assume a roll acceleration limit of 0.7 rad/sec².

**Rate of change of angular acceleration** (roll jerk rate), is another quantity for which we do not know of any generally accepted criterion. However, the roles of jerk rate limitation are to avoid inducing suspension oscillation and to avoid subjecting passengers to the sensation of jerk. These goals are
achieved if acceleration changes are made gradually such that the change from zero to the maximum allowed value takes about one second. Thus for our examples we will use a value of

\[ 0.7 \text{ rad/sec}^3 \]

The above roll acceleration and jerk limits must be converted to a per ft square and per ft cubed basis (via factors of vehicle speed) prior to their use in the formulae of table 1.

As already stated, we use \( f \) and \( h \) rather than \( W \) and \( A \) as inputs for controlling spiral geometry. The resulting values of \( W \) and \( A \) are then checked to insure that they do not exceed the acceptable limits.

8. GEOMETRY FOR CURVE LAYOUT

For purposes of curve layout, we use coordinate axes and sign conventions as follows:

For spiral joining tangent to curve,
- \( x \) axis is tangent itself
- \( y \) axis is normal to tangent passing through center of curve
- radius of curve is taken to be positive

For spiral joining two curves both of which have non-zero curvature,
- \( y \) axis passes through center of each curve
- \( x \) axis is tangent to (a geometrical extension of) the curve whose radius is larger in magnitude (or to either one if the curves reverse and the radii are of equal magnitude)
- radius of curve not tangent to \( x \) axis is taken to be positive, and radius of curve tangent to \( x \) axis is taken to be negative if the curves reverse and positive otherwise.

The features of a spiral which are used to determine overall curve layout are:

- The offset, defined as \( y \) coordinate of the point where (a continuation of) the curve with smaller (or equal) magnitude radius would intersect the \( y \)-axis.
- The bearing angle at the end of the spiral where it meets the smaller radius curve, measured upward from the \( x \)-axis.
- The starting point of the spiral specified by its \( x \) coordinate for case of spiral from tangent to curve and specified by its bearing angle relative to the \( x \) axis in case of spiral from curve to curve.

The overall layout of a curve placed symmetrically between two tangents is determined by the following items:

- total angle through which the track direction turns
- the curvature of the constant radius part of the curve
- the angle of turn within each spiral, and
- the x and y coordinates of the end of one spiral relative to axes with origin at the start of that spiral.

Additional information is usually desired as follows,

- arc length of each spiral is needed for accumulating track length along the route.
- coordinates of points along the spiral are needed for surveying or plotting the spiral.

A layout for a spiral joining a tangent to a curve is shown in Figure 3. Layouts for spirals between reverse and same sense curves are illustrated in Figures 4 and 5.

**Figure 3. Layout for spiral between tangent and curve**
Figure 4. Layout for spiral between reverse curves

FIGURE 4. Spiral between reverse curves.
Figure 5. Layout for spiral between curves of same sense
In this section we look first at new type spirals of minimum length. Exploratory calculations were done for the set of physical constraints

\begin{align*}
\text{track warp} & \leq 0.0007 \text{ radians/ft} \\
\text{vehicle roll acceleration} & \leq 0.7 \text{ radians/sec}^2 \\
\text{vehicle roll jerk} & \leq 0.7 \text{ radians/sec}^3
\end{align*}

and for speeds and curvatures in the ranges

\begin{align*}
60 \text{ mph} & \leq v \leq 210 \text{ mph} \\
0.5 & \leq \text{ degrees per 100 ft chord} \leq 4.0.
\end{align*}

These calculations have shown that

1) The above limit on roll acceleration is never reached for any combination of parameters in the above ranges and so has no effect on minimum transition length.

2) The shortness of a transition is limited by the roll jerk alone for curves of small degree. The roll motion that provides minimum transition length for such curves is of the form illustrated in Figure 6.

3) The shortness of a transition is limited both by roll jerk and by maximum allowed warp in the case of the tighter curves. For such curves, it was found that the shortest spiral is always obtained by eliminating the 2 zones of Figure 1 in which roll acceleration is a constant not equal to zero, and meeting the constraint on warp by adjusting the length of the middle zone in which warp is constant. This roll motion is of the form illustrated in Figure 7.
Figure 6. Roll motion for minimum length spirals which are not constrained by limit on track warp (typical for curves of small degree

Figure 7. Roll motion for minimum length spirals which are constrained by limit on track warp (and also for some spirals which have been made longer than the minimum possible
Basic features of some representative transitions of minimum length consistent with the constraints stated above are given in Table 4. In that table:

- The final roll is the roll required at the end of the spiral in order for the components of gravitational and centripetal acceleration in the plane of the car floor to cancel. Physical superelevation of rails will be less than that shown if full cancellation is not required or if some of the roll is provided by vehicle body "tilting".

- Const_w_frac is the fraction of the full roll center arc length in which the warp is constant. When this is zero, roll motion is as in Figure 6. When this is greater than zero, roll motion is as in Figure 7.

- The minimum turn angle is the total turn angle of a curve that consists just of two spirals back to back and has no region of constant curve radius. The total turn angle is generally a given quantity. Looking at the way that minimum turn angle values vary it is easy to see that for given total turn angle and desired curve negotiation speed, there will be a corresponding lower limit to the midpoint curve radius. Conversely, if total angle of turn and radius of constant curvature are both fixed then the range of speeds for which a solution exists will be subject to an upper limit.
Table 4. Minimum length new type spirals for representative values of curvature and train speed

curve negotiation at equilibrium
roll acceleration ramp rate = 0.7 rad/sec/sec/sec
max value of track warp = 0.0007 rad/ft
fraction of roll accel dist in which roll accel is constant = 0.0

<table>
<thead>
<tr>
<th>velocity mph</th>
<th>curvature deg</th>
<th>maximum roll ft</th>
<th>constant W deg</th>
<th>maximum W warp radians/ft</th>
<th>maximum acc radians/sec²</th>
<th>minimum turn ft</th>
<th>spiral length ft</th>
</tr>
</thead>
<tbody>
<tr>
<td>60 88</td>
<td>0.5 11459</td>
<td>1.20</td>
<td>0.640</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>11.55</td>
<td>290</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>2.52</td>
<td>0.069</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>1.12</td>
<td>112</td>
</tr>
<tr>
<td></td>
<td>2.0 2865</td>
<td>5.02</td>
<td>0.393</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>3.44</td>
<td>172</td>
</tr>
<tr>
<td></td>
<td>4.0 1433</td>
<td>9.94</td>
<td>0.640</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>11.55</td>
<td>290</td>
</tr>
<tr>
<td>90 132</td>
<td>0.5 11459</td>
<td>2.83</td>
<td>0.687</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>0.85</td>
<td>171</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>5.65</td>
<td>0.168</td>
<td>7.0E-4</td>
<td>0.254</td>
<td>2.30</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>2.0 2865</td>
<td>11.15</td>
<td>0.471</td>
<td>7.0E-4</td>
<td>0.254</td>
<td>7.22</td>
<td>362</td>
</tr>
<tr>
<td></td>
<td>4.0 1433</td>
<td>21.21</td>
<td>0.254</td>
<td>7.0E-4</td>
<td>0.254</td>
<td>24.01</td>
<td>612</td>
</tr>
<tr>
<td>120 176</td>
<td>0.5 11459</td>
<td>5.02</td>
<td>0.687</td>
<td>7.0E-4</td>
<td>0.208</td>
<td>1.38</td>
<td>275</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>9.94</td>
<td>0.234</td>
<td>7.0E-4</td>
<td>0.294</td>
<td>3.84</td>
<td>386</td>
</tr>
<tr>
<td></td>
<td>2.0 2865</td>
<td>19.11</td>
<td>0.517</td>
<td>7.0E-4</td>
<td>0.294</td>
<td>7.22</td>
<td>362</td>
</tr>
<tr>
<td></td>
<td>4.0 1433</td>
<td>33.46</td>
<td>0.254</td>
<td>7.0E-4</td>
<td>0.254</td>
<td>37.60</td>
<td>993</td>
</tr>
<tr>
<td>150 220</td>
<td>0.5 11459</td>
<td>7.81</td>
<td>0.687</td>
<td>7.0E-4</td>
<td>0.317</td>
<td>1.99</td>
<td>399</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>15.24</td>
<td>0.280</td>
<td>7.0E-4</td>
<td>0.328</td>
<td>5.69</td>
<td>573</td>
</tr>
<tr>
<td></td>
<td>2.0 2865</td>
<td>27.90</td>
<td>0.540</td>
<td>7.0E-4</td>
<td>0.328</td>
<td>17.40</td>
<td>897</td>
</tr>
<tr>
<td></td>
<td>4.0 1433</td>
<td>43.45</td>
<td>0.697</td>
<td>7.0E-4</td>
<td>0.328</td>
<td>48.63</td>
<td>1363</td>
</tr>
<tr>
<td>180 264</td>
<td>0.5 11459</td>
<td>11.15</td>
<td>0.687</td>
<td>7.0E-4</td>
<td>0.358</td>
<td>2.69</td>
<td>540</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>21.22</td>
<td>0.311</td>
<td>7.0E-4</td>
<td>0.360</td>
<td>7.75</td>
<td>788</td>
</tr>
<tr>
<td></td>
<td>2.0 2865</td>
<td>36.19</td>
<td>0.546</td>
<td>7.0E-4</td>
<td>0.360</td>
<td>22.50</td>
<td>1196</td>
</tr>
<tr>
<td>210 308</td>
<td>0.5 11459</td>
<td>14.95</td>
<td>0.026</td>
<td>7.0E-4</td>
<td>0.388</td>
<td>3.49</td>
<td>702</td>
</tr>
<tr>
<td></td>
<td>1.0 5730</td>
<td>27.46</td>
<td>0.330</td>
<td>7.0E-4</td>
<td>0.388</td>
<td>9.92</td>
<td>1021</td>
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<td></td>
<td>2.0 2865</td>
<td>43.03</td>
<td>0.540</td>
<td>7.0E-4</td>
<td>0.388</td>
<td>26.80</td>
<td>1484</td>
</tr>
</tbody>
</table>

* superelevation based on a 60 inch gage

Prior to gaining computational experience with the ways in which the proposed spirals vary as their defining parameters are varied, the author assumed that for fixed values of initial and final curvature and initial and final roll angle, independent variation of:
- roll acceleration ramp rate,
- fraction of acceleration distance in which acceleration is constant, and
- fraction of spiral length in which warp is constant
would allow independent variation of:
- offset between curves connected by the spiral,
- bearing angle at the start of the spiral (x coordinate for spiral starting from tangent), and
- bearing angle at the end of the spiral.

It soon became apparent that the Jacobean matrix consisting of derivatives of the three dependent variables with respect to the three independent variables is very close to zero and that only two of the
three dependent variables can be independently controlled. Moreover, if the offset is fixed, the range of variability of initial bearing angle is quite small. It has therefore been found convenient in practice to fix:
- fraction of acceleration distance in which acceleration is constant equal to zero and
- roll acceleration ramp rate equal to some constant such as 0.7 radians/sec$^3$,
and to vary
- fraction of spiral length in which warp is constant
when it is desired to achieve a prescribed value of offset.

As noted in Sections 2.3 and 11.1, when a new type spiral is being fitted to a reverse curve with prescribed offset, increase of roll axis height causes an increase in spiral length and a decrease in maximum track warp. Thus, roll axis height can be raised above 7 ft. in order to avoid excessive warp when offset is inadequate. Since this expedient sacrifices ride quality, it would be preferable to realign for adequate offset.

The results presented in Section 11 below were developed in this way. That is, offsets were calculated for the existing track geometry, and then spirals of the kind proposed here were found with matching offsets. The spirals found in this way join the curves at different points than do the existing traditional spirals. As will be seen from the examples in Section 11, it is generally not important where spirals of the proposed type meet curves to which they are fitted.

10. COMPARISON WITH TRADITIONAL RAILROAD SPIRALS

Traditional spiral design rules as applied in North American high-speed passenger practice are exemplified by section 3.3 of The Design Manual: Engineering (ref. 3) developed for the U.S. Northeast Corridor Improvement Project by De Leuw, Cather/Parsons. Further background may be found in the AREA Engineering Manual (ref. 4). The most notable feature of the traditional rules is that, conceptually, they cause the value of track warp (and therefore roll velocity of the vehicle wheel sets) to change abruptly from zero to its full value at or near the beginning of the spiral and abruptly back to zero at or near the end. From a basic mechanical point of view, such handling of the roll degree of freedom of the
vehicles is crude. (The traditional spiral’s ease of calculation was a virtue prior to the computer age but is now irrelevant).

The traditional rules for spiral length are summarized in Table 4.

**Table 5. Traditional rules for spiral length**

<table>
<thead>
<tr>
<th>superelevation of curve (inches)</th>
<th>minimum acceptable spiral length (ft)</th>
<th>track warp (rad/ft)</th>
<th>desired spiral length (ft)</th>
<th>track warp (rad/ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>166</td>
<td>2E-4</td>
<td>248</td>
<td>1.34E-4</td>
</tr>
<tr>
<td>3</td>
<td>249</td>
<td>2E-4</td>
<td>372</td>
<td>1.34E-4</td>
</tr>
<tr>
<td>4</td>
<td>332</td>
<td>2E-4</td>
<td>496</td>
<td>1.34E-4</td>
</tr>
<tr>
<td>5</td>
<td>415</td>
<td>2E-4</td>
<td>620</td>
<td>1.34E-4</td>
</tr>
<tr>
<td>6</td>
<td>498</td>
<td>2E-4</td>
<td>744</td>
<td>1.34E-4</td>
</tr>
</tbody>
</table>

As explained in Section 7 above, we believe that when roll motion is treated as proposed herein, the maximum allowable value of track warp can be increased from 2e-4 to about 7e-4 radians/ft.

In addition to the above rules for spiral length, traditional practice as documented in the aforementioned source requires that spirals for adjacent reverse curves be separated by tangent track of length 100 ft or 2.2 * maximum speed (mph), whichever is larger. It is possible that that rule had value for facilitating maintenance of track alignment prior to the availability of modern surveying instruments. However, from the point of view of vehicle dynamics, there is no basis for such a rule. Adjacent reverse curves can best be connected by a continuous spiral based on continuous change in vehicle roll angle as described here. When the adjacent curves curve in opposite directions, (i.e. are reverse curves) the spiral has zero curvature at the point where the roll angle (superelevation) passes through zero. Examples of spirals of the proposed type for connecting reverse curves are given in Section 11.1.
Referring to values for spirals to 5" superelevation in Table 4 (new spirals) and Table 5 (traditional spirals) one finds the spiral lengths shown in Table 6.

**Table 6. Traditional & proposed spiral lengths for 5" superelevation**

<table>
<thead>
<tr>
<th>design approach</th>
<th>spiral length (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>traditional minimum acceptable spiral length</td>
<td>415</td>
</tr>
<tr>
<td>new 2 deg curve, balanced for 60 mph</td>
<td>172</td>
</tr>
<tr>
<td>traditional desired spiral length</td>
<td>620</td>
</tr>
<tr>
<td>new 0.5 deg curve, balanced for 120 mph</td>
<td>275</td>
</tr>
</tbody>
</table>

In Table 6, the speeds associated with the new spirals would be higher if, as has traditionally been the case, trains traversed curves with some unbalance. It is evident that for given change in superelevation, the proposed spiral geometry allows spiral length to be reduced to about half the length which would be required with traditional spirals.

Further comparisons between the traditional and proposed spirals can be made with the help of plots showing both types of spiral in plan view. Plots of this type are given in Figures 8 through 13.

### 11. CALCULATIONS FOR NEC CURVES NUMBERED 308 & 309 (REVERSE) AND 340 (SIMPLE) WITH A GOAL OF PROVIDING FOR HIGHER SPEEDS

This section presents six examples of application of the new spiral design method to two specific curve situations on the AMTRAK route between New York City and Washington, D.C. The first three examples show that the new method can be used to eliminate a speed restriction to which conventional
and Metroliner passenger trains are presently subject. The last three examples show that the new method will allow redesign of existing curves for greatly increased speeds without requiring acquisition of any new land for right-of-way.

11.1 Reverse Curves 308 & 309 near Sharon Hill, PA

Curves 308 & 309 are adjacent reverse curves near Sharon Hill south of Philadelphia. Together with curve 311, they cause authorized speed on the high-speed tracks to drop from the surrounding value of 100 mph down to 80 mph. (In this section “current” refers to conditions in effect in 1985. There may have been some alterations in the mean time.) The restriction arises because there is not enough room between these reverse curves to develop any more than 3 in. superelevation based on traditional spiral and warp criteria.

These curves have features that in the case of track 2 are as follows:

Table 7. Nominal parameters for Curves 308 & 309a on track 2

<table>
<thead>
<tr>
<th>curve number</th>
<th>308</th>
<th>309a</th>
</tr>
</thead>
<tbody>
<tr>
<td>curve radius (ft)</td>
<td>5414.40</td>
<td>5720.0</td>
</tr>
<tr>
<td>curvature – d:m:s</td>
<td>1° 03' 29.6''</td>
<td>1° 0' 06.1''</td>
</tr>
<tr>
<td>curvature – degrees</td>
<td>1.0582261°</td>
<td>1.0016872°</td>
</tr>
<tr>
<td>length of spiral (ft)</td>
<td>117</td>
<td>132</td>
</tr>
<tr>
<td>existing superelevation</td>
<td>3''(approx)</td>
<td>3''(approx)</td>
</tr>
<tr>
<td>current speed limits</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>desired speed limits</td>
<td>100*</td>
<td>100*</td>
</tr>
<tr>
<td>desired superelevation</td>
<td>6''</td>
<td>6''</td>
</tr>
</tbody>
</table>

Chainage values along the current alignment and lengths of segments are as follows:

<table>
<thead>
<tr>
<th>at</th>
<th>chainage</th>
<th>length</th>
<th>segment</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC 308</td>
<td>295 + 18.29</td>
<td>132.0</td>
<td>308 spiral</td>
</tr>
<tr>
<td>TS 308</td>
<td>294 + 01.29</td>
<td>75.63</td>
<td>tangent between spirals</td>
</tr>
<tr>
<td>ST 309a</td>
<td>293 + 25.66</td>
<td>117.0</td>
<td>309a spiral</td>
</tr>
<tr>
<td>CS 309a</td>
<td>291 + 93.66</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*assuming superelevation also raised on curve 311
It may be noted that the above spiral and intervening tangent lengths are short of the values prescribed by traditional design rules as set forth in Section 9 above. In cases such as this, the superelevation runoff is commonly extended into the intervening tangent, which in this case is just long enough to allow compliance with the traditional minimum runoff distance.

Figures 8 through 11 all include a plot of the traditional spiral-tangent-spiral sequence which would connect curves 308 & 309a based on the above data and without reflecting superelevation runoff in the interior tangent that is likely to be present in the actual track. (How close the existing track comes to the traditional geometry will need to be checked based on measurement of the track.)

Each of Figures 8 through 11 also includes a plot of a spiral of the new type that can also connect those two curves.

Before discussing the Figures individually, we will point out some features that they share in common. Each figure has an upper area, a middle area, and a lower area. The upper area depicts curvature, the middle area depicts track alignment in plan view as seen from above, and the lower area depicts superelevation. The three areas display their quantities relative to a common x-axis defined as illustrated in Figure 4 or Figure 3 above. (Distances along the track are approximately equal to x-coordinate values, particularly near x = 0.) The lower area of each figure also depicts the transverse distance that the rails would be moved (sometimes referred to as the track throw) if the alignment were changed from traditional to the new alignment illustrated. Note that in the middle area the alignment is represented by the track offset from the base line (which is the x-axis). In each area, the y-axis scale has been made different from the common x-axis scale as an aide to visual comparison. Curves representing attributes of the existing spiral alignment can be recognized by their kinks and by their shorter length compared to curves representing the new alignment. If the constant curvature (or tangent) trackage on either side of the
transition were illustrated it would attach to the ends of the transition curves that are illustrated. Numerical parameter values underlying each Figure are given in the Table with the same number.

The new type spirals shown in Figures 8 through 10 are all based on a balancing speed of 90 mph and superelevation of 6 inches. These three figures illustrate how the new type spirals are affected by changes in roll axis height and roll jerk rate. They also show that each of the new type spirals has an alignment that is very close to the present alignment but that each is longer than the existing spiral-tangent-spiral sequence. Apart from possible increased clearance requirements due to increased superelevation, the lateral displacement between the existing and new type alignments is nowhere as much as 2 inches. The slight asymmetry between the traditional and new type alignments is due to the difference in length of the two adjacent traditional spirals and the slight difference in the radii of the two curves. Of the three new type spirals shown, that of Figure 10 appears to the author to provide the most desirable features for redesign of the reverse curves.

The encouraging conclusion to be drawn from these results is that it appears that it will be possible with very little expense to redesign reverse curves 308 & 309 to provide 6 inches superelevation and thereby eliminate the existing 20 mph speed reduction. In order to translate these concepts into practice, it will be necessary to develop a detailed design that includes verification that lateral shift of the tracks plus increased clearance requirements due to increased superelevation will not infringe clearances relative to fixed objects such as catenary poles and bridge structures.
Figure 8. Reverse curves 308 & 309, new spiral with 6 in. superelevation, 0 ft roll axis height.
Table 8 a. Parameters of traditional spiral of Figures 8 through 11

prescribed length of 1st spiral = 132.000000
prescribed length of tangent between rev spirals = 75.630000
prescribed length of 2nd spiral = 117.000000

bearing angle change in transition = -0.042052 degrees
transition end coords in transition axes
x2s = 324.615987 ft, y2s = -2.816613 ft

computed offset between curve arcs = 2.030645 ft.
arc length of transition = 324.630000 feet

computed transition start bearing in curve axes = 1.690806 degrees
transition end bearing angle in curve axes = 1.648754 degrees

Table 8 b. Parameters of new spiral of Figure 8

prescribed offset between arcs of the 2 curves = 2.030681 ft

vehicle speed = 132.000000 ft/sec = 90.000000 mph
balancing elevation of 1st curve = -5.656057 in.
balancing elevation of 2nd curve = 5.972217 in.

height of roll axis = 0.000000 ft
prescribed roll jerk rate = 0.700000 rad/sec/sec/sec

tolerance for convergence of offset = 0.000010

b1c below denotes bearing angle (radians) at start of spiral in curve axes
(axes with y axis through centers of the two curves and x axis tangent
to extension of larger radius curve)

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final fraction of total distance in which warp is const = 0.624664
max track warp utilized = 0.000535 rad/ft
max roll acceleration utilized = 0.222309 rad/sec/sec
bearing angle change in spiral = 0.125640 degrees
Figure 9. Reverse curves 308 & 309, new spiral with 6 in. superelevation, 7 ft roll axis height
Table 9. Parameters of new spiral of Figure 9

prescribed offset between arcs of the 2 curves = 2.030681 ft

vehicle speed = 132.000000 ft/sec = 90.000000 mph
balancing elevation of 1st curve = -5.656703 in.
balancing elevation of 2nd curve = 5.972978 in.

height of roll axis = 7.000000 ft
prescribed roll jerk rate = 0.700000 rad/sec/sec/sec

tolerance for convergence of offset = 0.000010

b1c below denotes bearing angle (radians) at start of spiral in curve axes
(axes with y axis through centers of the two curves and x axis tangent
to extension of larger radius curve)

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final fraction of total distance in which warp is const = 0.730515
max track warp utilized = 0.000411 rad/ft
max roll acceleration utilized = 0.194928 rad/sec/sec
bearing angle change in spiral = 0.153447 degrees
Figure 10. Reverse curves 308 & 309, like Figure 9 but with reduced roll jerk rate
Table 10. Parameters of new spiral of Figure 10

prescribed offset between arcs of the 2 curves = 2.030681 ft

vehicle speed = 132.000000 ft/sec = 90.000000 mph
balancing elevation of 1st curve = -5.656703 in.
balancing elevation of 2nd curve = 5.972978 in.

height of roll axis = 7.000000 ft
prescribed roll jerk rate = 0.400000 rad/sec/sec/sec

tolerance for convergence of offset = 0.000010

b1c below denotes bearing angle (radians) at start of spiral in curve axes
(axes with y axis through centers of the two curves and x axis tangent
to extension of larger radius curve)

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final fraction of total distance in which warp is const = 0.656395
max track warp utilized = 0.000413 rad/ft
max roll acceleration utilized = 0.147694 rad/sec/sec
bearing angle change in spiral = 0.159606 degrees

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Figure 11 is similar to Figures 8, 9, and 10 but shows a new type spiral designed to allow curves 308 & 309 to be superelevated to 15 inches for traversal at a balancing speed of 150 mph. Such a design would not be contemplated except as a part of the design of a completely new set of very high-speed tracks dedicated to a new service in which signaling would allow trains to proceed into highly elevated curves only when they were clear to proceed all the way through the curves at or near design speed. Whether such a system would be practical is open to question. The conclusion to be drawn from the Figure is that curves of the kind needed for such a service can be accommodated within the existing right-of-way.
The calculation underlying the new spiral of Figure 11 was done with the roll axis height raised above 7 ft. in order to lengthen the spiral and thereby keep the warp from exceeding the limit of 7e-4 radians/ft. When this example was done with a roll axis height of 7 ft. the new type spiral was somewhat shorter and the maximum value of warp was 0.00090 radians/ft., which, as discussed in Section 7, is presumably a little too high. With the roll axis height raised to 14 ft. as shown, the maximum warp is reduced to 0.00069 radians/ft. The corresponding maximum roll acceleration is 0.33 radians/sec/sec, which would cause passengers to experience lateral acceleration (toward the inside of the first curve and then toward the inside of the second curve) amounting to \(0.033 \times 7 = 2.3\) ft./sec/sec. This is 1.6 mph/sec and is thus comparable to the longitudinal acceleration normally experienced during braking. However, redesign of curves 308 & 309 for traversal at 150 mph would be contemplated only in connection with construction of a new pair of dedicated tracks. Therefore, the practical solution would be to increase the offset between the two reverse curves from 2 ft. to a value between 5 and 14 ft. so that a satisfactory value of maximum warp would be obtained with a 7 ft. roll axis height. (Offset of 5 ft. corresponds to roll jerk = 0.7 radians/sec/sec/sec and max warp = 0.0007 radians/ft. Offset of 14 ft. corresponds to roll jerk = 0.5 radians/sec/sec/sec and max warp = 0.0005 radians/ft.) A 14 ft. offset would be preferable, but keeping the tracks within the right of way might dictate a lesser figure.
Figure 11. REVERSE curves 308 & 309, NEW spiral with superel. = 16 in. & roll axis height = 14 ft.

Traditional spirals AND intervening TANGENT EXTEND from -169 ft to 156 ft.

Exploratory dynamic spiral extends from -487 ft to 486 ft. Maximum track throw to dynamic spiral = 14.0 in.

Louis T. Klauder Jr., PE
Table 11. Parameters of new spiral of Figure 11

prescribed offset between arcs of the 2 curves = 2.030681 ft

vehicle speed = 220.000000 ft/sec = 150.000001 mph
balancing elevation of 1st curve = -15.271324 in.
balancing elevation of 2nd curve = 16.073996 in.

height of roll axis = 14.000000 ft
prescribed roll jerk rate = 0.700000 rad/sec/sec/sec

tolerance for convergence of offset = 0.000010

b1c below denotes bearing angle (radians) at start of spiral in curve axes
(axes with y axis through centers of the two curves and x axis tangent to extension of larger radius curve)

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final fraction of total distance in which warp is const = 0.579577
max track warp utilized = 0.000688 rad/ft
max roll acceleration utilized = 0.325440 rad/sec/sec
bearing angle change in spiral = 0.265446 degrees

11.2 Ordinary curve 340 near Elkton, MD

Figure 12 is analogous to Figure 11 except that it pertains to a simple 1 degree curve and that the new type spiral shown on it is designed for a curve traversal at balancing speeds of 180 mph. Data underlying Figure 12 is given in tables 12a and 12b. (This example does not mimic existing curve 340 exactly in that it uses a traditional spiral of 500 ft. whereas curve 340 presently has spirals whose nominal lengths are 498 ft. and 598 ft. The point of the example is to treat a typical 1-degree curve.)

It is interesting to note in Figure 12 that with the vehicle roll center at a height of 7 feet above top of rail, the first motion to become visible as spiral traversal begins is the movement of the rails toward the
outside of the upcoming curve. This reflects the vehicle’s roll about its roll center before the latter has begun to move noticeably toward the inside of the curve.

Based on the results shown in Figure 12, we conclude that a 1-degree curve with traditional spirals that permits traversal at 110 mph can be redesigned and super-elevated for a balancing speed of 180 mph without requiring any additional right-of-way. (The superelevation illustrated in that figure does not fall within established practice. It could not be contemplated without provisions to insure, as a matter of safety, that trains would traverse it only at the intended speed. It would likely be wiser to reduce the curvature.)
FIGURE 12. Curve 340, exploratory spiral with superelevation = 21 in. & roll axis height = 7 ft.

Traditional spirals AND intervening TANGENT EXTEND from -250 ft to 250 ft.
Exploratory dynamic spiral extends from -495 ft to 478 ft. Maximum track throw to dynamic spiral = 4.3 in.

Louls T. Klauder Jr., PE
Table 12 a. Parameters of Traditional spiral of Figure 12

prescribed length of spiral = 500.000000

bearing angle change in transition = 2.499968 degrees
transition end coords in transition axes
  x2s = 499.904818 ft, y2s = 7.271124 ft

computed offset between curve arcs = 1.817905 ft.
arc length of transition = 500.000000 feet

computed transition start x in curve axes = -249.984136 ft
transition end bearing angle in curve axes = 2.499968 degrees

------------------------------------------------------------------------

Table 12 b. Parameters of new spiral of Figure 12

prescribed offset from tangent to arc of the curve = 1.817905 ft

vehicle speed = 264.000000 ft/sec = 180.000001 mph
balancing elevation of 1st curve = 0.000000 in.
balancing elevation of 2nd curve = 21.228826 in.

height of roll axis = 7.000000 ft
prescribed roll jerk rate = 0.700000 rad/sec/sec/sec

tolerance for convergence of offset = 0.000010

x1c below denotes x coordinate at start of spiral relative to
line of tangent track as x axis and y axis through center of curve

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final fraction of total distance in which warp is const = 0.536968

max track warp utilized = 0.000483 rad/ft

max roll acceleration utilized = 0.298858 rad/sec/sec

bearing angle change in spiral = 4.784861 degrees

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12. CONCLUSION

Traditional spiral transition design begins by looking at the spiral as a geometrical shape and is satisfied with making that shape “smooth”. In contrast to the traditional approach, the spiral design method proposed here begins by looking at vehicle roll motion and manages that motion in an effective manner. As a result, motion of passenger vehicles traversing spirals of the proposed type will be more comfortable at the same speeds and/or equally comfortable at higher speeds in comparison to motion of passenger vehicles traversing traditional spirals. In this respect, the proposed spiral design method may be said to correct a conceptual defect of the traditional method.

Comparing a traditional spiral and a spiral of the proposed type when both of them develop the same change of curvature and superelevation and provide about the same ride quality, the spiral of the proposed type can be expected to be about half the length of the traditional spiral.

One illustration of the benefit of the spiral design method proposed here is that it will allow several existing reverse curves on the AMTRAK route between New York City and Washington, D.C. to be redesigned with increased superelevation and with very little lateral shifting of the tracks. This will permit the elimination or easing of several speed restrictions that presently impose costs in running time and energy consumption. In the case of reverse curves numbered 308 and 309 south of Philadelphia, the superelevation can be raised from 3 inches to 6 inches with less than 2 inches lateral shift of the tracks related to the proposed new spiral geometry per se.

Another situation in which the proposed spiral design method can be a key to progress arises when there is a desire to take a narrow right-of-way that originally carried freight trains and upgrade it for use by passenger trains. In such a situation, the right-of-way property boundaries often make it impossible with
traditional spiral design rules to achieve the superelevations desired in order for passenger trip times to be competitive. Here also the proposed spiral design method will allow alignments that stay inside property boundaries to have higher superelevations.

13. ACKNOWLEDGEMENTS

Funding for a part of this work was provided by the Federal Railroad Administration (F.R.A)of the U.S. D.O.T. The substance of this paper was submitted to the F.R.A. in September of 1985 in an unpublished report entitled “Report of a study of Improved Spiral Geometry for High-speed Passenger Railroads”.

REFERENCES


